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Radicals of skew polynomial rings and skew Laurent polynomial rings

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ABSTRACT

We first introduce the σ -Wedderburn radical and the σ -Levitzki radical of a ring R , where σ is an automorphism of R . Using the properties of these radicals, we study the Wedderburn radical of the skew polynomial ring $R[x; \sigma]$ and the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$, and next observe the Levitzki radical of $R[x; \sigma]$ and $R[x, x^{-1}; \sigma]$. Furthermore we characterize the upper nilradical of $R[x; \sigma]$ and $R[x, x^{-1}; \sigma]$, via the upper σ -nil radical of R .

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Throughout this note, R denotes an associative ring with identity, and σ denotes an automorphism of R . Let $W(R)$, $P(R)$, $L(R)$, $N(R)$ and $J(R)$ denote the Wedderburn radical, the prime radical, the Levitzki radical, the upper nilradical, and the Jacobson radical of R , respectively. We use \mathbb{Z} to denote the ring of integers. We refer to [9] for basic facts of skew polynomial rings and skew Laurent polynomial rings.

The study of radicals of skew polynomial rings in relation to σ -radicals has occurred often over multiple decades. The prime radical of skew polynomial rings is being analyzed by Pearson and Stephenson [11]. They proved $P(R[x; \sigma]) = (P(R) \cap P_\sigma(R)) + P_\sigma(R)xR[x; \sigma]$, where $P_\sigma(R)$ is the σ -prime radical of R which was introduced in [11]. For the Jacobson radical of skew polynomial ring, Bedi and Ram [1] proved $J(R[x; \sigma]) = (I \cap J(R)) + IxR[x; \sigma]$, and moreover, if σ is of locally finite order then $J(R[x; \sigma]) = I[x; \sigma]$, where $I = \{r \in R \mid rx \in J(R[x; \sigma])\}$. On the other hand, Pearson, Stephenson and Watters introduced other radicals like as the σ -nil radical, the σ -Jacobson radical, the σ -Kleinfeld radical, and the σ -Brown-McCoy radical, and then show that $R[x; \sigma]$ is a σ -Jacobson

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ring if and only if R is a σ -Jacobson ring in [12]. The concept of the upper σ -nil radical was also introduced in [7] which is the same concept as σ -nil radicals of [12] by [7, Proposition 3.7].

For the continuation of radicals of skew polynomial rings, in this paper, we first introduce other σ -radicals analogous to the Wedderburn radical and the Levitzki radical of rings. Using the properties of these radicals, we study the Wedderburn radical of $R[x; \sigma]$ and $R[x, x^{-1}; \sigma]$, and then we observe the Levitzki radical, and the upper nilradical of $R[x; \sigma]$ and $R[x, x^{-1}; \sigma]$. In Section 1, the definitions of the σ -Wedderburn radical, the σ -Levitzki radical and related ideals are formulated. In Section 2, we study the σ -radicals of $R[X]$ induced by σ -radicals of R . Then, in Section 3, we characterize the Wedderburn radical, the Levitzki radical, and the upper nilradical of $R[x; \sigma]$ and $R[x, x^{-1}; \sigma]$ via related σ -nil radicals of R .

1. Radicals induced by an automorphism

We begin by giving a series of definitions with the aim of producing generalizations of classical radicals. Let R be a ring and I an ideal of R . If $\sigma(I) \subseteq I$, then I is called σ -ideal. If $\sigma(I) = I$, then I is called σ -invariant. The following definitions are found in [7]. An element a of R is called σ -nilpotent if for any integer $l \geq 1$, there exists a positive integer $m = m(l)$, depending on l , such that $a\sigma^l(a)\sigma^{2l}(a) \cdots \sigma^{ml}(a) = 0$. A subset S of R is called σ -nil if every element in S is σ -nilpotent. Due to Lam, Leroy and Matcsuk [7], the upper σ -nil radical of R , denote by $N_\sigma(R)$, is given by $N_\sigma(R) = \sum \{I \mid I \text{ is } \sigma\text{-nil } \sigma\text{-ideal of } R\}$. A subset S of R is called σ - n -nil ($n \geq 1$) if every element in S is σ^m -nilpotent for any $m \geq n$. Pearson, Stephenson and Watters [12] also introduced the σ -nil radical of R by the sum of all σ -invariant σ - n -nil ideals for some $n \geq 1$. Note that the upper σ -nil radical is equal to the σ -nil radical by [7, Proposition 3.7].

We now introduce other σ -radicals analogous to the Wedderburn radical and the Levitzki radical of a ring R .

Definition 1.1.

- (1) A subset S of R is called σ -nilpotent if for any integer $l \geq 1$, there exists a positive integer $m = m(l)$ such that $S\sigma^l(S)\sigma^{2l}(S) \cdots \sigma^{ml}(S) = 0$.
- (2) A subset S of R is called locally σ -nilpotent if every finite subset of S is σ -nilpotent.
- (3) The σ -Wedderburn radical of R , denoted by $W_\sigma(R)$, is given by

$$W_\sigma(R) = \sum \{I \mid I \text{ is a } \sigma\text{-nilpotent } \sigma\text{-ideal of } R\}.$$

- (4) The σ -Levitzki radical of R , denoted by $L_\sigma(R)$, is given by

$$L_\sigma(R) = \sum \{I \mid I \text{ is a locally } \sigma\text{-nilpotent } \sigma\text{-ideal of } R\}.$$

According to Pearson and Stephenson [11], a proper σ -ideal P of R is σ -prime if whenever $AB \subseteq P$ for an ideal A and a σ -ideal B , we have that either $A \subseteq P$ or $B \subseteq P$. If in addition P is σ -invariant, then P is called strongly σ -prime; a proper σ -ideal Q of R is σ -semiprime if for any ideal A and an integer m such that $A\sigma^n(A) \subseteq Q$ for all $n \geq m$, we have $A \subseteq Q$. Notice that any σ -prime ideal is σ -semiprime. Pearson and Stephenson [11] defined the σ -prime radical of R , denoted by $P_\sigma(R)$, as the intersection of all strongly σ -prime ideals of R .

We have the following lemma with the help of [7, Lemma 2.2, Theorem 4.21] and [11].

Lemma 1.2. For a proper σ -invariant ideal P of R , the following conditions are equivalent:

- (1) P is σ -prime.
- (2) For $a, b \in R$, if for some positive integer m , $aR\sigma^n(b) \subseteq P$ for all $n \geq m$ then $a \in P$ or $b \in P$.
- (3) For $a, b \in R$, if for some integer m , $aR\sigma^n(b) \subseteq P$ for all $n \geq m$ then $a \in P$ or $b \in P$.

Proof. It is enough to show $(1) \Leftrightarrow (2)$. Let P be a σ -prime ideal of R . For $a, b \in R$, suppose $aR\sigma^n(b) \subseteq P$ for all integers $n \geq m$, where m is a positive integer. Consider the ideal $\sum_{n=m}^{\infty} R\sigma^n(b)R$. Then $\sum_{n=m}^{\infty} R\sigma^n(b)R$ is a σ -ideal of R , and moreover $(RaR)(\sum_{n=m}^{\infty} R\sigma^n(b)R) \subseteq P$. Since P is σ -prime, $a \in P$ or $\sigma^n(b) \in P$ for all $n \geq m$, and therefore $a \in P$ or $b \in P$ since P is σ -invariant.

Conversely, let I and J be ideals of R with J a σ -ideal. Suppose $IJ \subseteq P$. For any $b \in J$, we have $\sigma^n(b) \in J$ for all positive integer n . Thus $aR\sigma^n(b) \subseteq IJ \subseteq P$ for any $a \in I$. By hypothesis, $a \in P$ or $b \in P$, and therefore $I \subseteq P$ or $J \subseteq P$. \square

Remark. Note that a σ -ideal P of R is σ -semiprime if and only if for $a \in R$, if for some integer m , $aR\sigma^n(a) \subseteq P$ for all $n \geq m$ then $a \in P$ by the same method as in the proof of Lemma 1.2.

Due to [3], an element $a \in R$ is called *strongly σ -nilpotent* if for any sequence $(t_n)_{n=0}^{\infty}$ of positive integers such that $t_{n+1} \geq 1 + \sum_{i=0}^n t_i$, and for any sequence $(a_n)_{n=0}^{\infty}$ such that $a_0 = a$ and $a_{n+1} \in a_n R \sigma^{t_n}(a_n)$ for all $n \geq 0$, there is an integer m such that $a_m = 0$. The following results can be founded in [3].

Proposition 1.3. $P_{\sigma}(R) = \{a \in R \mid a \text{ is strongly } \sigma\text{-nilpotent}\}$ and $P_{\sigma}(R)$ is a σ -nil σ -invariant ideal of R . Moreover, $P_{\sigma}(R)$ is the smallest σ -semiprime ideal of R .

Pearson, Stephenson and Watters proved $P_{\sigma}(R) \subseteq N_{\sigma}(R) \subseteq J_{\sigma}(R)$, where $J_{\sigma}(R)$ denotes the σ -Jacobson radical defined by the intersection of all the σ -primitive ideals of R [12, Proposition 3.9]. We now prove the inclusions $W_{\sigma}(R) \subseteq P_{\sigma}(R) \subseteq L_{\sigma}(R) \subseteq N_{\sigma}(R)$.

Lemma 1.4.

- (1) Let I be a σ -ideal of R . Then I is σ -nilpotent if and only if for any integer $l \geq 1$, there exists a positive integer N , not depending on l , such that $I\sigma^l(I)\sigma^{2l}(I) \cdots \sigma^{Nl}(I) = 0$. Actually, the N is the index in the case $l = 1$.
- (2) If I and J are σ -nilpotent σ -ideals of R , then $I + J$ is σ -nilpotent.

Proof. (1) Suppose that I is σ -nilpotent. Then $I\sigma(I)\sigma^2(I) \cdots \sigma^N(I) = 0$ for some $N \geq 1$. Since I is a σ -ideal, for any integer $l \geq 1$, $I\sigma^l(I)\sigma^{2l}(I) \cdots \sigma^{Nl}(I) \subseteq I\sigma(I)\sigma^2(I) \cdots \sigma^N(I) = 0$, completing the proof.

(2) By (1), for any integer $l \geq 1$, there exist positive integers N_1 and N_2 , not depending on l , such that $I\sigma^l(I) \cdots \sigma^{N_1 l}(I) = 0$ and $J\sigma^l(J) \cdots \sigma^{N_2 l}(J) = 0$, respectively. Put $N = N_1 + N_2$. Then

$$(I + J)\sigma^l(I + J) \cdots \sigma^{Nl}(I + J) = I\sigma^l(I) \cdots \sigma^{Nl}(I) + \cdots + J\sigma^l(J) \cdots \sigma^{Nl}(J).$$

The number of I or J occurring in each term of this equality is $\geq N_1$ or $\geq N_2$. Using the fact that I and J are σ -ideals, each term of the equality is contained in $I\sigma^l(I) \cdots \sigma^{N_1 l}(I)$ or $J\sigma^l(J) \cdots \sigma^{N_2 l}(J)$. Thus $(I + J)\sigma^l(I + J) \cdots \sigma^{Nl}(I + J) = 0$, and therefore $I + J$ is σ -nilpotent. \square

Proposition 1.5. $W_{\sigma}(R) \subseteq P_{\sigma}(R)$.

Proof. Let $a \in W_{\sigma}(R)$. Assume that $a \notin P_{\sigma}(R)$. Then there exists a σ -prime ideal P with $a \notin P$. Since $a \in W_{\sigma}(R)$, $a \in I$ for some σ -nilpotent σ -ideal I of R . Then by Lemma 1.4(1), for any integer $l \geq 1$, there exists a positive integer n , not depending on l , such that $I\sigma^l(I)\sigma^{2l}(I) \cdots \sigma^{nl}(I) = \{0\} \subseteq P$. Since $I \not\subseteq P$, $\sigma^l(I)\sigma^{2l}(I) \cdots \sigma^{nl}(I) \subseteq P$. Since P is σ -invariant, $I\sigma^l(I) \cdots \sigma^{(n-1)l}(I) \subseteq P$. Continuing this process, we have $a \in I \subseteq P$, a contradiction. \square

The following example shows that $W_{\sigma}(R) \subsetneq P_{\sigma}(R)$ for an automorphism σ of R in general.

Example 1.6. Let K be an infinite field and $K[\{t_i\}_{i \in \mathbb{Z}}]$ the polynomial ring over K , and $J = \langle \{t_{n_1} t_{n_2} t_{n_3} \mid n_3 - n_2 \geq (n_2 - n_1)^2\} \rangle$ be the ideal of $K[\{t_i\}_{i \in \mathbb{Z}}]$, modifying the construction of Ram [13]. Define $R = K[\{t_i\}_{i \in \mathbb{Z}}]/J$. The K -automorphism σ of $K[\{t_i\}_{i \in \mathbb{Z}}]$ defined by sending each t_i to t_{i+1} induces an automorphism σ on R . Let $I = \sum_{i=-\infty}^{\infty} \sigma^i(t_1)R$ and $f = \sum_{\text{finite}} t_{i_1} t_{i_2} \cdots t_{i_{j_1}} \in I$. Put

$$D = \{|i_u - i_v| \mid i_u, i_v \text{ are indices occurring in a monomial of } f\}$$

and $k = \max(D)$. Then we have

$$f \sigma^l(f) \sigma^{2l}(f) \cdots \sigma^{(k+l)^2 l}(f) = 0$$

for any integer $l \geq 1$. Fix $l \geq 1$. We claim that f is strongly nilpotent. Suppose by way of contradiction that f is not strongly nilpotent, that is $f \notin P_\sigma(R)$. Put $f = f_0$. Then $f_0 \notin P$ for some σ -prime ideal P of R . Thus by Lemma 1.2, there is an integer $s_0 \geq 1$ and $r_0 \in R$ such that $f_0 r_0 \sigma^{s_0}(f_0) \notin P$. Letting $f_1 = f_0 r_0 \sigma^{s_0}(f_0)$, we get $f_2 = f_1 r_1 \sigma^{s_1}(f_1) \notin P$ for some $r_1 \in R$ and $s_1 \geq 1 + s_0$. Then $f_2 = f_0 \sigma^{s_0}(f_0) \sigma^{s_0+s_1}(f_0) r_2$ for some $r_2 \in R$. Continuing this process, we have sequences $(f_m)_{m=0}^\infty$ in R and $(s_m)_{m=0}^\infty$ of positive integers such that $s_{m+1} \geq 1 + \sum_{i=0}^m s_i$ and $f_{m+1} = f_m r_m \sigma^{s_m}(f_m)$ with $f_m \notin P$ for all $m \geq 0$, and so $f_{m+1} = f_0 \sigma^{s_0}(f_0) \sigma^{s_0+s_1}(f_0) \cdots \sigma^{s_0+s_1+\cdots+s_m}(f_0) r_{m+1} \notin P$ for some $r_{m+1} \in R$ for all $m \geq 0$, but it is impossible because there exists an integer N such that $\sum_{i=0}^N s_i \geq (k+l)^2 l$. Clearly, I is a σ -ideal of R . Therefore $I \subseteq P_\sigma(R)$ by Proposition 1.3, entailing $t_1 \in P_\sigma(R)$.

Next we will show $I \not\subseteq W_\sigma(R)$. Let $J = \sum_{i=0}^\infty \sigma^i(t_1)R \subseteq I$ and suppose $t_1 \in W_\sigma(R)$. Then, by Lemma 3.1 to follow, $J \sigma(J) \sigma^2(J) \cdots \sigma^k(J) = 0$ for some integer $k \geq 1$. Thus

$$0 = t_1 \sigma(t_k) \sigma^2(t_{2k}) \cdots \sigma^k(t_{k^2}) = t_1 t_{k+1} t_{2k+2} \cdots t_{k^2+k}.$$

But $(k^2 + k) - (k + 1) = k^2 - 1 < k^2$, a contradiction. This yields $t_1 \notin W_\sigma(R)$ by Lemma 3.1 to follow.

Next we prove that $P_\sigma(R) \subseteq L_\sigma(R)$ by using the following lemma.

Lemma 1.7.

- (1) Let I and J be locally σ -nilpotent ideals of R with J a σ -ideal. Then $I + J$ is a locally σ -nilpotent ideal of R .
- (2) Let I and J be locally σ -nilpotent right ideals of R with $\sigma(I) \subseteq I$ and $\sigma(J) \subseteq J$. Then RI and $I + J$ are locally σ -nilpotent σ -ideals.
- (3) $L_\sigma(R)$ is the largest locally σ -nilpotent σ -ideal of R .

Proof. (1) We refer the proof of [7, Proposition 3.4]. Let $C = \{a_1 + b_1, a_2 + b_2, \dots, a_n + b_n\}$ be a subset of $I + J$, where $a_i \in I$ and $b_j \in J$. Let $l \geq 1$. Since I is locally σ -nilpotent, there exists a positive integer $k = k(l)$ such that $A \sigma^l(A) \cdots \sigma^{kl}(A) = 0$, where $A = \{a_1, \dots, a_n\}$. Then since J is a σ -ideal,

$$(a_{i_1} + b_{i_1}) \sigma^l(a_{i_2} + b_{i_2}) \cdots \sigma^{kl}(a_{i_{k+1}} + b_{i_{k+1}}) = a_{i_1} \sigma^l(a_{i_2}) \cdots \sigma^{kl}(a_{i_{k+1}}) + \alpha = \alpha$$

for some $\alpha \in J$, where $a_{i_j} \in A$. Let B be the set of all such α 's. Then B is a finite subset of J since l and k are fixed. Since J is locally σ -nilpotent, there exists a positive integer t such that $B \sigma^{(k+1)l}(B) \cdots \sigma^{t(k+1)l}(B) = 0$. Then

$$\begin{aligned} & (a_{i_1} + b_{i_1}) \sigma^l(a_{i_2} + b_{i_2}) \cdots \sigma^{kl}(a_{i_{k+1}} + b_{i_{k+1}}) \sigma^{(k+1)l}(a_{i_{k+2}} + b_{i_{k+2}}) \cdots \sigma^{Nl}(a_{i_{(t+1)k+t+1}} + b_{i_{(t+1)k+t+1}}) \\ &= \alpha_1 \sigma^{(k+1)l}((a_{i_{k+2}} + b_{i_{k+2}}) \sigma^l(a_{i_{k+3}} + b_{i_{k+3}}) \cdots \sigma^{kl}(a_{i_{2k+2}} + b_{i_{2k+2}})) \cdots \end{aligned}$$

$$\begin{aligned}
&= \alpha_1 \sigma^{(k+1)l} (\alpha_2) \sigma^{2(k+1)l} ((a_{i_{2k+3}} + b_{i_{2k+3}}) \sigma^l (a_{i_{2k+4}} + b_{i_{2k+4}}) \cdots \sigma^{kl} (a_{i_{3k+3}} + b_{i_{3k+3}})) \cdots \\
&\vdots \\
&= \alpha_1 \sigma^{(k+1)l} (\alpha_2) \sigma^{2(k+1)l} (\alpha_3) \cdots \sigma^{t(k+1)l} (\alpha_{t+1}) \in B \sigma^{(k+1)l} (B) \sigma^{2(k+1)l} (B) \cdots \sigma^{t(k+1)l} (B) = 0
\end{aligned}$$

where $N = t(k+1) + k$ and $\alpha_1, \dots, \alpha_{t+1} \in B$. This yields

$$C \sigma^l(C) \cdots \sigma^{Nl}(C) = 0.$$

(2) Let $S = \{\sum_{\text{finite}} r_i a_i \mid a_i \in I, r_i \in R\}$ be a finite subset of RI . Fix an integer $l \geq 1$. Let $S' = \{a_i r_i \mid \sum_{\text{finite}} r_i a_i \in S\}$. Then S' is a finite subset of I , and so $S' \sigma^l(S') \sigma^{2l}(S') \cdots \sigma^{ml}(S') = 0$ for some positive integer m . Thus S is also σ -nilpotent.

By the preceding argument, RI and RJ are locally σ -nilpotent and so is $RI + RJ$ by (1). Thus $I + J \subseteq RI + RJ$ is locally σ -nilpotent.

(3) It follows from (1). \square

Proposition 1.8. $P_\sigma(R) \subseteq L_\sigma(R)$.

Proof. By Proposition 1.3, $P_\sigma(R)$ is the smallest σ -semiprime ideal. So it suffices to show that $L_\sigma(R)$ is σ -semiprime. Note that $L_\sigma(R)$ is a σ -ideal by the definition of it. Assume on the contrary that $L_\sigma(R)$ is not σ -semiprime. Then there exists $a \in R \setminus L_\sigma(R)$ such that $aR\sigma^n(a) \subseteq L_\sigma(R)$ for all $n \geq m$ for some integer m . Since $a \notin L_\sigma(R)$, there exists an integer $k \geq 1$ such that

$$a\sigma^k(a)\sigma^{2k}(a) \cdots \sigma^{sk}(a) \neq 0 \quad \text{for all } s \geq 1. \quad (1)$$

But $aR\sigma^n(a) \subseteq L_\sigma(R)$, and so $a\sigma^k(a)\sigma^{2k}(a) \cdots \sigma^{s_1 k}(a) \in L_\sigma(R)$ for some $s_1 k \geq m$. Thus, from the non-equality (1), we get a non-stationary sequence

$$\begin{aligned}
&0 \neq a\sigma^k(a)\sigma^{2k}(a) \cdots \sigma^{s_1 k}(a), \\
&0 \neq a\sigma^k(a) \cdots \sigma^{s_1 k}(a) \sigma^{(s_1+1)k} (a\sigma^k(a) \cdots \sigma^{s_1 k}(a)), \\
&0 \neq a\sigma^k(a) \cdots \sigma^{s_1 k}(a) \sigma^{(s_1+1)k} (a\sigma^k(a) \cdots \sigma^{s_1 k}(a)) \sigma^{2(s_1+1)k} (a\sigma^k(a) \cdots \sigma^{s_1 k}(a)), \\
&\vdots \\
&0 \neq a\sigma^k(a) \cdots \sigma^{s_1 k}(a) \sigma^{(s_1+1)k} (a\sigma^k(a) \cdots \sigma^{s_1 k}(a)) \cdots \sigma^{t(s_1+1)k} (a\sigma^k(a) \cdots \sigma^{s_1 k}(a))
\end{aligned}$$

for all $t \geq 1$. But $L_\sigma(R)$ is locally σ -nilpotent by Lemma 1.7(3). Then since $a\sigma^k(a)\sigma^{2k}(a) \cdots \sigma^{s_1 k}(a) \in L_\sigma(R)$, we have

$$a\sigma^k(a) \cdots \sigma^{s_1 k}(a) \sigma^{(s_1+1)k} (a\sigma^k(a) \cdots \sigma^{s_1 k}(a)) \cdots \sigma^{t_1(s_1+1)k} (a\sigma^k(a) \cdots \sigma^{s_1 k}(a)) = 0$$

for some $t_1 \geq 1$. This induces a contradiction. \square

In the following example we use the result: for positive integers k and m , let $G(k, m)$ denote the least integer such that if $g \geq G(k, m)$ and if $A = (a_n)_{n=0}^{g-1}$ is a strictly increasing sequence of integers with bounded gap $a_n - a_{n-1} \leq m$, $1 \leq n \leq g-1$, then A contains a k -term arithmetic progression. The number of $G(k, m)$ does exist [10]. The existence of $G(k, m)$ is an easy consequence of Van der Waerden's theorem [4,15].

Using this result, we show that $P_\sigma(R) \subsetneq L_\sigma(R)$ for an automorphism σ of R as follows.

Example 1.9. Let K be an infinite field and $K[\{t_i\}_{i \in \mathbb{Z}}]$ the polynomial ring over K , and $J = \langle \{t_{n_1} t_{n_2} t_{n_3} \mid n_3 - n_2 = n_2 - n_1 > 0\} \rangle$ be the ideal of $K[\{t_i\}_{i \in \mathbb{Z}}]$, according to the construction of Ram [13]. Define $R = K[\{t_i\}_{i \in \mathbb{Z}}]/J$. The K -automorphism σ of $K[\{t_i\}_{i \in \mathbb{Z}}]$ defined by sending each t_i to t_{i+1} induces an automorphism σ on R . Ram proved that the skew polynomial ring $R[x; \sigma]$ is prime [13, Example 3.2(ii)]. Thus $P_\sigma(R) = 0$ by [11, Corollary 1.4].

Consider the σ -ideal $I = \sum_{i=0}^{\infty} \sigma^i(t_1)R$ of R . Let S be a finite subset of I . Then $S \subseteq \sum_{i=1}^k t_i R$, after reordering if necessary, where $k \geq 3$. We claim that for any integer $l \geq 1$, there exists a positive integer $N = N(l)$ such that $S\sigma^l(S)\sigma^{2l}(S)\cdots\sigma^{Nl}(S) = 0$. For this, it suffices to show $t_{i_0}\sigma^l(t_{i_1})\sigma^{2l}(t_{i_2})\cdots\sigma^{Nl}(t_{i_n}) = 0$ for any $i_j \in \{1, 2, \dots, k\}$. Consider the increasing sequence

$$i_{j_0} < i_{j_1} + k^2l < i_{j_2} + 2k^2l < \cdots < i_{j_n} + nk^2l$$

with bounded gap $k^2l + k$, where $i_{j_s} \in \{1, 2, \dots, k\}$ and $n \geq G(3, k^2l + k)$. Then by [10], the sequence contains an arithmetic progression of length 3. Therefore $t_1 \in L_\sigma(R)$ by Lemma 3.9 to follow, completing the proof.

To show that $L_\sigma(R) \subsetneq N_\sigma(R)$ for an automorphism σ of R , we introduce the Morse semigroup G generated by two elements a and b satisfying the relations $U^3 = 0$ for every product U of a 's and b 's, constructed as follows: consider the following sequence of a 's and b 's

$$a \mid b \mid ba \mid baab \mid baababba \mid \cdots$$

Hedlund and Morse [5] proved that there is no block of terms U such that the block UUU occurs in the sequence.

By the definition of $L_\sigma(R)$, it is clear that $L_\sigma(R) \subseteq N_\sigma(R)$. We now show that $L_\sigma(R) \subsetneq N_\sigma(R)$ for an automorphism σ of R .

Example 1.10. We refer an example in [14]. Let K be a countable field and \bar{A} be the algebra of polynomials with zero constant terms in noncommuting indeterminates t_1, t_2, t_3 over K . Then the elements in \bar{A} can be enumerated, i.e., $\bar{A} = \{a_1, a_2, \dots\}$. Let $I = \langle \{a_1^{3^{\deg a_1}}, a_2^{3^{\deg a_2}}, \dots\} \rangle$ be an ideal of \bar{A} generated by the subset $\{a_1^{3^{\deg a_1}}, a_2^{3^{\deg a_2}}, \dots\}$, where $\deg a_i$ denotes the degree of the polynomial a_i . Put $R_0 = \bar{A}/I$ and $R = K + R_0$. The K -automorphism σ of \bar{A} defined by $t_1 \mapsto t_2, t_2 \mapsto t_3$ and $t_3 \mapsto t_1$ induces an automorphism σ on R .

We first claim $L_\sigma(R) = 0$. Note that $L_\sigma(R) \subseteq N_\sigma(R) \subseteq R_0$. Assume $0 \neq f \in L_\sigma(R)$ and take two distinct elements a, b in $RfR \subseteq L_\sigma(R)$. Then by the result of Hedlund and Morse [5], we can choose a subset $\alpha_0, \alpha_1, \dots, \alpha_k$ in $\{a, b\}$ such that $\alpha_0\alpha_1\cdots\alpha_k \neq 0$. Let $B = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$. Since $B \subseteq RfR \subseteq L_\sigma(R)$, there exists a positive integer m such that $B\sigma^3(B)\cdots\sigma^{3m}(B) = 0$. In case $m < k$, we get $\alpha_0\alpha_1\cdots\alpha_k = 0$, a contradiction. In case $m > k$, we can also take α_i 's in $\{a, b\}$ such that $\alpha_0\alpha_1\cdots\alpha_k\alpha_{i_1}\cdots\alpha_{i_t} \neq 0$, where $t \geq m - k$. Then $t + k \geq m$ and so $\alpha_0\alpha_1\cdots\alpha_k\alpha_{i_1}\cdots\alpha_{i_t} = 0$, a contradiction. Therefore $L_\sigma(R) = 0$.

Next we claim that $N_\sigma(R) \neq 0$. Actually we show $N_\sigma(R) = R_0$. Let $f \in R_0$ and $l(\geq 1)$ be an integer. If $l = 3k$ for some positive integer k , then $f\sigma^l(f)\sigma^{2l}(f)\cdots\sigma^{3^{\deg f}l}(f) = f^{3^{\deg f}-1} = 0$. Suppose $l \neq 3k$ and let $g = f\sigma^l(f)\sigma^{2l}(f)$. Then

$$f\sigma^l(f)\sigma^{2l}(f)\sigma^{3l}(f)\sigma^{4l}(f)\sigma^{5l}(f)\cdots = f\sigma^l(f)\sigma^{2l}(f)f\sigma^l(f)\sigma^{2l}(f)\cdots$$

Since $g^{3^{\deg g}} = 0$, $f\sigma^l(f)\sigma^{2l}(f)\cdots\sigma^{3^{\deg g}l}(f) = 0$. Therefore $f \in N_\sigma(R)$, and consequently $N_\sigma(R) = R_0$.

2. σ -Radicals of polynomial rings

Let X be the set of commuting indeterminates. If σ is an automorphism of R then σ induces an automorphism, still denoted by σ , on $R[X]$, by $\sigma(a_0 + \sum_{j=1}^n a_j I_j) = \sigma(a_0) + \sum_{j=1}^n \sigma(a_j) I_j$, where each I_j is a finite product of indeterminates in X . If J is a σ -ideal of R then σ also induces an endomorphism, still denoted by σ , on R/J , by $\sigma(a + J) = \sigma(a) + J$. If $\sigma(J) = J$, then σ induces an automorphism.

Lemma 2.1.

- (1) $L_\sigma(R/L_\sigma(R)) = 0$.
- (2) If $L_\sigma(R[X]) \neq 0$ then $L_\sigma(R) \neq 0$.

Proof. (1) Note that $L_\sigma(R)$ is a σ -invariant ideal of R by Lemma 3.9 to follow. Let $\bar{R} = R/L_\sigma(R)$ and \bar{I} a locally σ -nilpotent σ -ideal in \bar{R} . We claim that I is locally σ -nilpotent. Note that I is a σ -ideal. Let S be a finite subset of I . Then $\bar{S} \subseteq \bar{I}$, and so, for a fixed integer $l \geq 1$, there exists a positive integer $t = t(l)$ such that $\bar{S}\sigma^l(\bar{S}) \cdots \sigma^{tl}(\bar{S}) = \bar{0}$, and hence $S\sigma^l(S) \cdots \sigma^{tl}(S) \subseteq L_\sigma(R)$. Letting $T = S\sigma^l(S) \cdots \sigma^{tl}(S)$ then T is a finite subset of $L_\sigma(R)$. By Lemma 1.7(3), $L_\sigma(R)$ is locally σ -nilpotent, and so for $(t+1)l$, there exists a positive integer n such that $T\sigma^{(t+1)l}(T) \cdots \sigma^{n(t+1)l}(T) = 0$. Therefore $I \subseteq L_\sigma(R)$.

(2) Let σ be an automorphism of $R[X]$ induced by σ . Let I be a nonzero locally σ -nilpotent σ -ideal of $R[X]$. Let X_1 be the set of all indeterminates occurring in all polynomials in I . Let $\emptyset \neq X_0$ be a finite subset of X_1 . Set A be the set of all polynomials in I which are contained in $R[X_0]$. Note $R[X_0] = R[Y_0][x]$ for each $x \in X_0$ and $Y_0 = X_0 \setminus \{x\}$. Let $S = R[Y_0]$ so that $R[X_0] = S[x]$. Let J be the ideal of $R[X_0]$ generated by A . Since I is a σ -ideal, so is J .

Next let K be the ideal of S generated by leading coefficients of polynomials in J . Then K is a nonzero σ -ideal of S since J is a σ -ideal of $R[X_0]$. Let $\{a_{m_1}, a_{m_2}, \dots, a_{m_n}\}$ be a finite subset of K . Then there exists a finite subset $\{f_1, f_2, \dots, f_n\} \subseteq J$ such that a_{m_i} is the leading coefficient of f_i for each i . Since J is locally σ -nilpotent, for any integer $l \geq 1$, there exists a positive integer t such that $f_{i_1}\sigma^l(f_{i_2}) \cdots \sigma^{tl}(f_{i_t}) = 0$, where $i_j \in \{1, 2, \dots, n\}$. Thus $a_{m_{i_1}}\sigma^l(a_{m_{i_2}}) \cdots \sigma^{tl}(a_{m_{i_t}}) = 0$. Therefore K is locally σ -nilpotent σ -ideal of S , and hence $L_\sigma(R[Y_0]) \neq 0$.

Now let H be a nonzero locally σ -nilpotent σ -ideal of $R[Y_0]$, and then we proceed the above process. Continuing this process, we get $L_\sigma(R) \neq 0$. \square

Theorem 2.2. $L_\sigma(R[X]) = L_\sigma(R)[X]$.

Proof. Note that $(R/L_\sigma(R))[X] \cong R[X]/L_\sigma(R)[X]$. By Lemma 2.1(1), $L_\sigma(R[X]/L_\sigma(R)[X]) = 0$, and so $L_\sigma(R[X]) \subseteq L_\sigma(R)[X]$. To show the reverse inclusion, let I be a locally σ -nilpotent σ -ideal of R . We claim that $I[X] \subseteq L_\sigma(R[X])$. Let $S = \{f_1, f_2, \dots, f_n\} \subseteq I[X]$, and let A be the set of all coefficients of polynomials in S . Then A is a finite subset of I . Since I is locally σ -nilpotent, for any integer $l \geq 1$, there exists a positive integer $t = t(l)$ such that $A\sigma^l(A) \cdots \sigma^{tl}(A) = 0$. This implies that $S\sigma^l(S) \cdots \sigma^{tl}(S) = 0$, and so S is σ -nilpotent. Thus $I[X] \subseteq L_\sigma(R[X])$, and therefore $L_\sigma(R)[X] \subseteq L_\sigma(R[X])$, completing the proof. \square

Moreover, by the same methods as in the proofs of Lemma 2.1 and Theorem 2.2, we have the following.

Theorem 2.3. $W_\sigma(R[X]) = W_\sigma(R)[X]$.

Lemma 2.4. A ring R is σ -prime (resp. σ -semiprime) if and only if $R[X]$ is σ -prime (resp. σ -semiprime).

Proof. We apply the well-known method for the (semi)prime case as in the proof of [6, Proposition 10.18]. Let $S = R[X]$ be σ -prime, and for some $m \in \mathbb{Z}$ suppose $aR\sigma^n(b) = 0$ for all $n \geq m$ where $a, b \in R$. Then $aS\sigma^n(b) = aR[X]\sigma^n(b) = 0$ and so $a = 0$ or $b = 0$, obtaining that R is σ -prime.

For the converse suppose that R is σ -prime, and for some $m \in \mathbb{Z}$ let $fS\sigma^n(g) = 0$ for all $n \geq m$ where $f, g \in S$. Note that there is a finite subset X_0 of X such that $f, g \in R[X_0]$, concluding $fR[X_0]\sigma^n(g) = 0$. Thus it suffices to consider the case of X being finite, and moreover the induction enables the computation for $R[x]$ for a single indeterminate x to complete the proof. Let a and b be the leading coefficients of f and g , respectively. Then $fR[x]\sigma^n(g) = 0$ implies that $aR\sigma^n(g) = 0$ for all $n \geq m$; hence we get $a = 0$ or $b = 0$. Thus $f = 0$ or $g = 0$.

The semiprime case can be proved when $a = b$ and $f = g$. \square

Theorem 2.5. $P_\sigma(R[X]) = P_\sigma(R)[X]$.

Proof. We apply the well-known method for the (semi)prime case as in the proof of [6, Proposition 10.19]. Since $R/P_\sigma(R)$ is σ -semiprime, $R[X]/P_\sigma(R)[X] (\cong (R/P_\sigma(R))[X])$ is σ -semiprime by Lemma 2.4, i.e., $P_\sigma(R)[X]$ is σ -semiprime ideal of $R[X]$. Thus $P_\sigma(R[X]) \subseteq P_\sigma(R)[X]$ by Proposition 1.3.

Conversely, it suffices to show $P_\sigma(R)[X] \subseteq Q$ for any σ -prime ideal Q of $R[X]$. First we claim that $Q \cap R$ is a σ -prime ideal in R . For some $m \in \mathbb{Z}$ suppose $aR\sigma^n(b) \subseteq Q \cap R$ for all $n \geq m$ where $a, b \in R$. Then

$$aR[X]\sigma^n(b) = aR\sigma^n(b)[X] \subseteq Q$$

and hence we get $a \in Q$ or $b \in Q$. It is immediate that $P_\sigma(R) \subseteq Q \cap R \subseteq Q$, and consequently $P_\sigma(R)[X] \subseteq Q$, completing the proof. \square

3. Wedderburn, Levitzki and upper nil radicals of skew polynomial rings and skew Laurent polynomial rings

In this section, we observe the Wedderburn radical, the Levitzki radical and the upper nil radical of skew polynomial rings and skew Laurent polynomial rings via related σ -radicals.

Lemma 3.1.

- (1) $W_\sigma(R)$ is a σ -invariant ideal of R .
- (2)

$$\begin{aligned} W_\sigma(R) &= \left\{ a \in R \mid \sum_{i=0}^{\infty} R\sigma^i(a)R \text{ is } \sigma\text{-nilpotent} \right\} \\ &= \left\{ a \in R \mid \sum_{i=m}^{\infty} R\sigma^i(a)R \text{ is } \sigma\text{-nilpotent for every } m \in \mathbb{Z} \right\} \\ &= \left\{ a \in R \mid \sum_{i=n}^{\infty} R\sigma^i(a)R \text{ is } \sigma\text{-nilpotent for some } n \leq 0 \right\} \\ &= \left\{ a \in R \mid \sum_{i=0}^{\infty} \sigma^i(a)R \text{ is } \sigma\text{-nilpotent} \right\}. \end{aligned}$$

- (3) If R is left or right Noetherian then $W_\sigma(R) = \{a \in R \mid \sum_{i=-\infty}^{\infty} R\sigma^i(a)R \text{ is } \sigma\text{-nilpotent}\}$.

Proof. It is obvious that K is σ -nilpotent if and only if so is $\sigma^{-1}(K)$ if and only if so is $\sigma(K)$ for an ideal K of R . We will use this freely.

(1) Since $\sigma(W_\sigma(R)) \subseteq W_\sigma(R)$, $W_\sigma(R) \subseteq \sigma^{-1}(W_\sigma(R))$. For the reverse inclusion, let $\sigma^{-1}(a) \in \sigma^{-1}(W_\sigma(R))$. Then $a \in \sum_{\text{finite}} I_j$, where each I_j is a σ -nilpotent σ -ideal of R . Since I_j is a σ -ideal, $I_j \subseteq \sigma^{-1}(I_j)$ and so $\sigma(\sigma^{-1}(I_j)) = I_j \subseteq \sigma^{-1}(I_j)$. This implies that $\sigma^{-1}(I_j)$ is a σ -ideal of R . Moreover $\sigma^{-1}(I_j)$ is also σ -nilpotent since I_j is σ -nilpotent. Thus $\sum_{\text{finite}} \sigma^{-1}(I_j) \subseteq W_\sigma(R)$ by Lemma 1.4(2), and hence $\sigma^{-1}(a) \in W_\sigma(R)$. This yields $\sigma^{-1}(W_\sigma(R)) \subseteq W_\sigma(R)$, entailing $W_\sigma(R) \subseteq \sigma(W_\sigma(R))$.

(2) Let $A = \{a \in R \mid \sum_{i=0}^{\infty} R\sigma^i(a)R \text{ is } \sigma\text{-nilpotent}\}$. Then, clearly $A \subseteq W_\sigma(R)$. For the reverse inclusion, let $w \in W_\sigma(R)$. Then there exist σ -nilpotent σ -ideals I_j in R such that $w \in \sum_{\text{finite}} I_j$. By Lemma 1.4(2), $\sum_{\text{finite}} I_j$ is also a σ -nilpotent σ -ideal of R . Thus $\sum_{i=0}^{\infty} R\sigma^i(w)R$ is σ -nilpotent since $\sum_{i=0}^{\infty} R\sigma^i(w)R \subseteq \sum_{\text{finite}} I_j$. Therefore $W_\sigma(R) \subseteq A$, entailing $W_\sigma(R) = A$.

To obtain the second expression of $W_\sigma(R)$, let $a \in W_\sigma(R)$. Then, by (1), $\sigma^{-j}(a) \in W_\sigma(R)$ for every positive integer j . So the σ -ideal $\sum_{i=0}^{\infty} R\sigma^i(\sigma^{-j}(a))R$ is σ -nilpotent by the first expression of $W_\sigma(R)$. This yields

$$W_\sigma(R) = \left\{ a \in R \mid \sum_{i=m}^{\infty} R\sigma^i(a)R \text{ is } \sigma\text{-nilpotent for every } m \in \mathbb{Z} \right\}.$$

Considering the including relation between the first and second expressions, we also obtain

$$W_\sigma(R) = \left\{ a \in R \mid \sum_{i=n}^{\infty} R\sigma^i(a)R \text{ is } \sigma\text{-nilpotent for some } n \leq 0 \right\}.$$

Finally, let $B = \{a \in R \mid \sum_{i=0}^{\infty} \sigma^i(a)R \text{ is } \sigma\text{-nilpotent}\}$. If $\sum_{i=0}^{\infty} \sigma^i(a)R$ is σ -nilpotent then so is $R(\sum_{i=0}^{\infty} \sigma^i(a)R) = \sum_{i=0}^{\infty} R\sigma^i(a)R$, entailing $a \in W_\sigma(R)$ and $B \subseteq W_\sigma(R)$. Conversely let $a \in W_\sigma(R)$. Then $\sum_{i=0}^{\infty} R\sigma^i(a)R$ is σ -nilpotent by the first expression of $W_\sigma(R)$, and so $\sum_{i=0}^{\infty} \sigma^i(a)R$ is σ -nilpotent, obtaining $B \supseteq W_\sigma(R)$.

(3) Let $C = \{a \in R \mid \sum_{i=-\infty}^{\infty} R\sigma^i(a)R \text{ is } \sigma\text{-nilpotent}\}$. If $\sum_{i=-\infty}^{\infty} R\sigma^i(a)R$ is σ -nilpotent then so is $\sum_{i=0}^{\infty} R\sigma^i(a)R$, entailing $a \in W_\sigma(R)$ and $C \subseteq W_\sigma(R)$. Conversely let $a \in W_\sigma(R)$. Then there exist σ -nilpotent σ -ideals I_j ($j \in J$ and $|J| < \infty$) in R such that $a \in \sum_{j \in J} I_j$. Since I_j is a σ -ideal, $I_j \subseteq \sigma^{-1}(I_j)$ and so $\sigma(\sigma^{-1}(I_j)) = I_j \subseteq \sigma^{-1}(I_j)$; hence $\sigma^{-1}(I_j)$ is a σ -ideal. Inductively we get that $\sigma^{-k}(I_j)$ is a σ -nilpotent σ -ideal of R for every positive integer k . Now we have $\sigma^{-k}(a) \in \sum_{\text{finite}} \sigma^{-k}(I_j)$ for every positive integer k , and so

$$\sum_{k=1}^{\infty} R\sigma^{-k}(a)R \subseteq \sum_{j \in J} \sum_{k=1}^{\infty} \sigma^{-k}(I_j).$$

Here we have an ascending chain of ideals

$$\sigma^{-1}(I_j) \subseteq \sigma^{-2}(I_j) \subseteq \cdots \subseteq \sigma^{-k}(I_j) \subseteq \cdots.$$

But if R is left or right Noetherian, there exists a positive integer N such that $\sigma^{-N}(I_j) = \sigma^{-(N+1)}(I_j) = \sigma^{-(N+2)}(I_j) = \cdots$. This yields $\sum_{k=1}^{\infty} \sigma^{-k}(I_j) = \sigma^{-N}(I_j)$ and

$$\sum_{j \in J} \sum_{k=1}^{\infty} \sigma^{-k}(I_j) = \sum_{j \in J} \sigma^{-N}(I_j).$$

By (1), we get $\sigma^{-N}(I_j) \subseteq W_\sigma(R)$ for all $j \in J$. Then Lemma 1.4(2) implies $\sum_{j \in J} \sigma^{-N}(I_j)$ is σ -nilpotent since every $\sigma^{-N}(I_j)$ is a σ -nilpotent σ -ideal of R . This entails that $\sum_{k=1}^{\infty} R\sigma^{-k}(a)R$ is also σ -nilpotent. From $a \in W_\sigma(R)$, we already have that $\sum_{i=0}^{\infty} R\sigma^i(a)R$ is σ -nilpotent by (2). Therefore

$$\sum_{i=-\infty}^{\infty} R\sigma^i(a)R = \sum_{i=0}^{\infty} R\sigma^i(a)R + \sum_{k=1}^{\infty} R\sigma^{-k}(a)R$$

is also σ -nilpotent by Lemma 1.4(2), entailing $a \in C$ and $C = W_\sigma(R)$. \square

Lemma 3.2. Let R be a right ideal I of R with $\sigma(I) \subseteq I$, and $n \geq 1$. Then I is σ -nilpotent if and only if I is σ^n -nilpotent.

Proof. Suppose that I is σ -nilpotent. For an integer $l \geq 1$, there exists a positive integer N such that $I\sigma^l(I)\sigma^{2l}(I)\cdots\sigma^{Nl}(I) = 0$. Since $\sigma(I) \subseteq I$, we also have

$$\begin{aligned} 0 &= I\sigma^l(I)\sigma^{2l}(I)\cdots\sigma^{Nl}(I) \\ &\supseteq I\sigma^l(\sigma^{(n-1)l}(I))\sigma^{2l}(\sigma^{(n-1)2l}(I))\cdots\sigma^{Nl}(\sigma^{(n-1)Nl}(I)) \\ &= I\sigma^{nl}(I)\sigma^{n2l}(I)\cdots\sigma^{nNl}(I), \end{aligned}$$

entailing that I is σ^n -nilpotent.

Conversely suppose that I is σ^n -nilpotent. For an integer $l \geq 1$, $I\sigma^l(I)\sigma^{2l}(I)\cdots\sigma^{(n-1)l}(I) \subseteq I$. Put $I' = I\sigma^l(I)\sigma^{2l}(I)\cdots\sigma^{(n-1)l}(I)$. Since I is σ^n -nilpotent and $I' \subseteq I$, there exists a positive integer t such that $I'\sigma^{nl}(I')\sigma^{2nl}(I')\cdots\sigma^{(tn+n-1)l}(I') = 0$. Replacing I' by $I\sigma^l(I)\sigma^{2l}(I)\cdots\sigma^{(n-1)l}(I)$, we obtain $I\sigma^l(I)\sigma^{2l}(I)\cdots\sigma^{(tn+n-1)l}(I) = 0$. \square

Let $x^{-1}a = bx^{-1}$ for $a, b \in R$. Then $a = bx^{-1} = \sigma(b)$ and so $b = \sigma^{-1}(a)$. Inductively $x^{-n}a = \sigma^{-n}(a)x^{-n}$ for any $n \geq 1$.

Theorem 3.3. $W(R[x, x^{-1}; \sigma]) = (W(R) \cap W_\sigma(R))[x, x^{-1}; \sigma] = W_\sigma(R)[x, x^{-1}; \sigma]$.

Proof. Let $S = R[x, x^{-1}; \sigma]$ and $f(x) = \sum_{i=m_1}^{m_2} a_i x^i \in W(S)$ with $m_1 \leq m_2 \in \mathbb{Z}$. Then $Sf(x)S$ is nilpotent and so $(Sf(x)S)^k = 0$ for some integer $k \geq 2$. Without loss of generality, we can put $f(x) = \sum_{i=0}^m a_i x^i$ with $m \geq 0$. Especially $(Rx^{\alpha_1}f(x))(Rx^{\alpha_2}f(x))\cdots(Rx^{\alpha_k}f(x)) = 0$ for every $\alpha_p \in \mathbb{Z}$ ($p = 1, 2, \dots, k$), and so

$$R\sigma^{t_0}(a_0)R\sigma^{t_1}(a_0)R\cdots R\sigma^{t_{k-1}}(a_0)R = 0 \quad (1)$$

for any integers t_i 's. Notice that every coefficient of every polynomial in $(Sa_0S)^k$ is contained in the left side of the equality (1). Thus $(Sa_0S)^k = 0$ and so $a_0 \in W(S)$ (hence $a_0 \in W_\sigma(R)$).

Here let $I = \sum_{i=0}^{\infty} R\sigma^i(a_0)R$. Then clearly I is a σ -ideal of R . Note that for any integer $l \geq 1$, every sum-factor of every element in $I\sigma^l(I)\sigma^{2l}(I)\cdots\sigma^{(k-1)l}(I)$ is of the form

$$r_{s_1}\sigma^{p_1}(a_0)r_{s_2}\sigma^{p_2}(a_0)r_{s_3}\cdots\sigma^{p_k}(a_0)r_{s_{k+1}} \quad (2)$$

for some r_{ij} 's in R and some nonnegative integers p_i 's. But the product (2) is contained in the left side of the equality (1), and so $I\sigma^l(I)\sigma^{2l}(I)\cdots\sigma^{(k-1)l}(I) = 0$. Thus I is a σ -nilpotent σ -ideal, entailing $a_0 \in W_\sigma(R)$ by Lemma 3.1, entailing $a_0 \in W_\sigma(R)$.

Next since $a_0 \in W(S)$ and $f(x) \in W(S)$, $a_1x + a_2x^2 + \cdots + a_mx^m \in W(S)$, and so $a_1 + a_2x + \cdots + a_mx^{m-1} \in W(S)$. Let $f_1(x) = a_1 + a_2x + \cdots + a_mx^{m-1} \in W(S)$, and repeat the above process. Then

we also get $a_1 \in W(R) \cap W_\sigma(R)$. Continuing this process, we get $a_0, a_1, \dots, a_m \in W(R) \cap W_\sigma(R)$. Therefore $f(x) \in (W(R) \cap W_\sigma(R))[x, x^{-1}; \sigma]$.

Conversely, let $g(x) = \sum_{i=m_1}^{m_2} b_i x^i \in W_\sigma(R)[x, x^{-1}; \sigma]$ with $m_1 \leq m_2 \in \mathbb{Z}$. We will show that $Sg(x)S$ is nilpotent, and so we can put $g(x) = \sum_{i=1}^m b_i x^i$ with $m \geq 1$ without loss of generality.

Set $K_i = \sum_{j=0}^{\infty} R\sigma^j(b_i)R$ for $i = 1, \dots, m$. Then the σ -ideals K_i 's are σ -nilpotent by Lemma 3.1. Next put $K = \sum_{i=1}^m K_i$. Then the σ -ideal K is also σ -nilpotent by Lemma 1.4(2). Moreover since K is a σ -ideal, Lemma 1.4(1) implies that there exists a positive integer N such that $K\sigma^s(K) \cdots \sigma^{sN}(K) = 0$ for all $s \geq 1$. Consider the product $(Sg(x)S)^{m(N+1)}$ and let P be a sum-factor of the expansion of $(\sum_{\text{finite}} h_j(x)g(x)k_j(x))^{m(N+1)} \in (Sg(x)S)^{m(N+1)}$. Then $b_i x^i$ must occur at least $(N+1)$ -times in P for some $i \in \{1, \dots, m\}$, and so P is contained in

$$(Kx^{j_0})K_i x^i (Kx^{j_1})K_i x^i (Kx^{j_2}) \cdots K_i x^i (Kx^{j_N})K_i x^i (Kx^{j_{N+1}})$$

where $j_t \in \mathbb{Z}$ for $t = 0, 1, \dots, N+1$. Notice that

$$\begin{aligned} & K_i x^i (Kx^{j_1})K_i x^i (Kx^{j_2}) \cdots K_i x^i (Kx^{j_N})K_i x^i (Kx^{j_{N+1}}) \\ &= K_i \sigma^i(K) x^{i+j_1} K_i \sigma^i(K) x^{i+j_2} \cdots K_i \sigma^i(K) x^{i+j_N} K_i \sigma^i(K) x^{i+j_{N+1}} \\ &\subseteq K_i \sigma^i(K) x^{i+j_1} \sigma^i(K) x^{i+j_2} \cdots \sigma^i(K) x^{i+j_N} \sigma^i(K) x^{i+j_{N+1}} \\ &= K_i \sigma^i(K) \sigma^{2i+j_1}(K) x^{2i+j_1+j_2} \cdots \sigma^i(K) x^{i+j_N} \sigma^i(K) x^{i+j_{N+1}} \\ &= K_i \sigma^i(K) \sigma^{2i+j_1}(K) \sigma^{3i+j_1+j_2}(K) \cdots \sigma^{Ni+\sum_{t=1}^{N-1} j_t}(K) \sigma^{(N+1)i+\sum_{t=1}^N j_t}(K) x^{Ni+\sum_{t=1}^{N+1} j_t} \\ &\subseteq K \sigma^i(K) \sigma^{2i+j_1}(K) \sigma^{3i+j_1+j_2}(K) \cdots \sigma^{(N+1)i+\sum_{t=1}^N j_t}(K) x^{Ni+\sum_{t=1}^{N+1} j_t} \\ &= K \sigma^i(K) \sigma^{2i}(\sigma^{j_1}(K)) \sigma^{3i}(\sigma^{j_1+j_2}(K)) \cdots \sigma^{(N+1)i}(\sigma^{\sum_{t=1}^N j_t}(K)) x^{Ni+\sum_{t=1}^{N+1} j_t}. \end{aligned} \quad (3)$$

Put $M = \sum_{t=1}^{N+1} |j_t|$. Then $M + \sum_{t=1}^k j_t \geq 0$ for all $k = 1, \dots, N+1$, and so the image of the coefficient of the term (3) under σ^M is as follows:

$$\begin{aligned} & \sigma^M(K \sigma^i(K) \sigma^{2i}(\sigma^{j_1}(K)) \sigma^{3i}(\sigma^{j_1+j_2}(K)) \cdots \sigma^{(N+1)i}(\sigma^{\sum_{t=1}^N j_t}(K))) \\ &= \sigma^M(K) \sigma^i(\sigma^M(K)) \sigma^{2i}(\sigma^{M+j_1}(K)) \sigma^{3i}(\sigma^{M+(j_1+j_2)}(K)) \cdots \sigma^{(N+1)i}(\sigma^{M+(\sum_{t=1}^N j_t)}(K)) \\ &\subseteq K \sigma^i(K) \sigma^{2i}(K) \cdots \sigma^{Ni}(K) \sigma^{(N+1)i}(K) = 0. \end{aligned}$$

But σ^M is also an automorphism, and so $K \sigma^i(K) \sigma^{2i}(\sigma^{j_1}(K)) \sigma^{3i}(\sigma^{j_1+j_2}(K)) \cdots \sigma^{(N+1)i}(\sigma^{\sum_{t=1}^N j_t}(K)) = 0$. This yields $(Kx^{j_0})K_i x^i (Kx^{j_1})K_i x^i (Kx^{j_2}) \cdots K_i x^i (Kx^{j_N})K_i x^i (Kx^{j_{N+1}}) = 0$, entailing $P = 0$. Thus $Sg(x)S$ is nilpotent and $g(x) \in W(S)$.

With these two proofs, we now obtain the equality $W(R[x, x^{-1}; \sigma]) = (W(R) \cap W_\sigma(R))[x, x^{-1}; \sigma] = W_\sigma(R)[x, x^{-1}; \sigma]$. \square

Corollary 3.4. $W_\sigma(R) \subseteq W(R)$.

Note. (1) If R is left or right Noetherian then $W_\sigma(R) = W(R)$.

(2) If R is left or right Noetherian then $W(R[x, x^{-1}; \sigma]) = W(R)[x, x^{-1}; \sigma] = W_\sigma(R)[x, x^{-1}; \sigma]$.

Proof. (1) It suffices to show $W_\sigma(R) \supseteq W(R)$ by Corollary 3.4. Let $a \in W(R)$. Then RaR is nilpotent and so $\sigma^i(RaR) = R\sigma^i(a)R$ is also nilpotent for any $i \in \mathbb{Z}$. Say $(RaR)^k = 0$ for some $k \geq 1$. This yields $(R\sigma^i(a)R)^k = 0$. If R is left or right Noetherian then

$$\sum_{i=-\infty}^{\infty} R\sigma^i(a)R = \sum_{i=-N}^N R\sigma^i(a)R$$

for some $N \geq 1$, considering the ascending chain

$$RaR \subseteq \sum_{i=-1}^1 R\sigma^i(a)R \subseteq \sum_{i=-2}^2 R\sigma^i(a)R \subseteq \dots$$

Then

$$\left(\sum_{i=-\infty}^{\infty} R\sigma^i(a)R \right)^{(2N+1)k} = \left(\sum_{i=-N}^N R\sigma^i(a)R \right)^{(2N+1)k} = 0,$$

and so for any $l \geq 1$ we have $I\sigma^l(I) \dots \sigma^{(2N+1)kl}(I) = 0$ where $I = \sum_{i=-\infty}^{\infty} R\sigma^i(a)R$. This implies that I is σ -nilpotent, and hence $a \in W_{\sigma}(R)$ by Lemma 3.1(3).

(2) It follows from Theorem 3.3 and (1). \square

An automorphism σ of R is of locally finite order if for every $r \in R$ there exists a positive integer $n = n(r)$ such that $\sigma^n(r) = r$.

Lemma 3.5. $W(R) = W_{\sigma}(R)$ when σ is of locally finite order.

Proof. Let $a \in W(R)$ and $\sigma^k(a) = a$ for some $k \geq 1$. Then $(RaR)^m = 0$ for some integer $m \geq 2$. Note that $\sigma^i(RaR) = R\sigma^i(a)R$ is nilpotent for each integer $i \geq 0$. Since $\sigma^k(a) = a$, $\sum_{i=0}^{\infty} R\sigma^i(a)R = \sum_{i=0}^{k-1} R\sigma^i(a)R$ is nilpotent and a σ -ideal. Put $A = \sum_{i=0}^{k-1} R\sigma^i(a)R$ and $A^t = 0$ for some integer $t \geq 2$. Thus for any integer $l \geq 1$, $A\sigma^l(A) \dots \sigma^{(t-1)l}(A) \subseteq A^t = 0$, and therefore $a \in W_{\sigma}(R)$. \square

Theorem 3.6. $W(R[x; \sigma]) = W(R)[x; \sigma]$ when σ is of locally finite order.

Proof. Let $S = R[x; \sigma]$ and $f(x) = \sum_{i=0}^m a_i x^i \in W(S)$. Then $Sf(x)S$ is nilpotent and so $(Sf(x)S)^k = 0$ for some integer $k \geq 2$. Especially $(Rx^{\alpha_1} f(x))(Rx^{\alpha_2} f(x)) \dots (Rx^{\alpha_k} f(x)) = 0$ for every $\alpha_p \geq 0$ ($p = 1, 2, \dots, k$), and we get

$$R\sigma^{t_0}(a_0)R\sigma^{t_1}(a_0)R \dots R\sigma^{t_{k-1}}(a_0)R = 0 \quad (1)$$

for any nonnegative integers t_i 's satisfying $t_0 \leq t_1 \leq \dots \leq t_{k-1}$. This implies $(R\sigma^i(a_0)R)^k = 0$ for all $i \geq 0$; hence $(\sum_{i=0}^s R\sigma^i(a_0)R)^{(s+1)k} = 0$ for any $s \geq 0$. Now let $\sigma^{s_0}(a_0) = a_0$ for some $s_0 \geq 1$, and let $I = \sum_{i=0}^{\infty} R\sigma^i(a_0)R$. Then $I = \sum_{i=0}^{s_0-1} R\sigma^i(a_0)R$, and I is nilpotent and $\sigma(I) = I$. Thus, for any integer $l \geq 1$, $I\sigma^l(I)\sigma^{2l}(I) \dots \sigma^{(s_0 k-1)l}(I) = I^{s_0 k} = 0$ and so I is a σ -nilpotent σ -invariant ideal, entailing $a_0 \in W_{\sigma}(R) = W(R)$ by Lemmas 3.1 and 3.5.

We here claim that $a_0 \in W(S)$. Let $g = g_0(x)a_0g_1(x)a_0g_2(x) \dots a_0g_{s_0 k}(x) \in (Sa_0S)^{s_0 k}$. Then each coefficient of this polynomial g is contained in $I^{s_0 k}$, and so g is zero. This implies that Sa_0S is nilpotent and so $a_0 \in W(S)$.

Next let $f_1(x) = f(x) - a_0 \in W(S)$ and so $(Sf_1(x)S)^q = 0$ for some integer $q \geq 2$. Thus we get

$$R\sigma^{t_0}(a_1)R\sigma^{1+t_1}(a_1)R \dots R\sigma^{q-1+t_{q-1}}(a_1)R = 0 \quad (2)$$

for any nonnegative integers t_i 's satisfying $t_0 \leq t_1 \leq \dots \leq t_{q-1}$. Let $\sigma^{s_1}(a_1) = a_1$ for some $s_1 \geq 1$, and let $J = \sum_{i=0}^{\infty} R\sigma^i(a_1)R$. Then clearly $J = \sum_{i=0}^{s_1-1} R\sigma^i(a_1)R$ and $\sigma(J) = J$. Note that for any integer $l \geq 1$, every sum-factor of every element in $J\sigma^l(J)\sigma^{2l}(J)\dots\sigma^{(q-1)l}(J) = J^q$ is of the form

$$r_{s_0}\sigma^{p_0}(a_1)r_{s_1}\sigma^{p_1}(a_1)r_{s_2}\dots r_{s_{q-1}}\sigma^{p_{q-1}}(a_1)r_{s_q}$$

for some r_{i_j} 's in R and some integers p_i 's, where $0 \leq p_i \leq s_i - 1$. Suppose that $p_i \geq p_j$ for $i < j$. Since $\sigma^{s_1}(a_1) = a_1$, we can take $\sigma^{p_j}(a_1) = \sigma^{ns_1+p_j}(a_1)$ such that $p_i < ns_1 + p_j$ for some $n \geq 1$. Then Eq. (2) implies $J\sigma^l(J)\sigma^{2l}(J)\dots\sigma^{(q-1)l}(J) = 0$. Thus J is a σ -nilpotent σ -invariant ideal, and so $a_1 \in W_\sigma(R) = W(R)$ by Lemmas 3.1 and 3.5. By the same method as above, we get $a_1 \in W(S)$. Consequently, we have $a_0, a_1, \dots, a_m \in W_\sigma(R)$ inductively. This yields $W(S) \subseteq W(R)[x; \sigma]$.

Conversely, let $g(x) = \sum_{i=0}^m b_i x^i \in W_\sigma(R)[x; \sigma]$. Let $\sigma^{s_i}(b_i) = b_i$ and $K_i = \sum_{j=0}^{s_i-1} R\sigma^j(b_i)R$ for $i = 0, 1, \dots, m$. Then each K_i is σ -invariant and so nilpotent. This implies that each $K_i[x; \sigma]$ is also a nilpotent ideal of S and so is $\sum_{i=0}^m K_i[x; \sigma]$. Therefore $g(x) \in \sum_{i=0}^m K_i[x; \sigma] \subseteq W(S)$. \square

In Theorem 3.6, the condition “ σ is of locally finite order” is essential by the following example.

Example 3.7. For $i \in \mathbb{Z}$, let $F_i = F$ be a field and put $K = \bigoplus_{i \in \mathbb{Z}} F_i$. Let $R = \langle \bigoplus_{i \in \mathbb{Z}} F_i, \langle 1 \rangle \rangle$ be the F -subalgebra of $\prod_{i=1}^{\infty} F_i$ generated by $\bigoplus_{i \in \mathbb{Z}} F_i$ and $\langle 1 \rangle$, where $\langle 1 \rangle$ is the identity of $\prod_{i \in \mathbb{Z}} F_i$. Let σ be an automorphism of R defined by $\sigma(\sum a_i) = \sum a_{i+1}$. Suppose that $0 \neq a = \sum a_i \in R$ is such that $a_i = 0$ for all $m \leq i \leq n$, where m, n denote fixed integers. Note that $(R[x; \sigma]axR[x; \sigma])^{n-m+2} = 0$ by [7, Example 4.5]. Hence $ax \in W(R[x; \sigma])$, but $ax \notin W(R)[x; \sigma] = W_\sigma(R)[x; \sigma]$, in fact, $W(R) = W_\sigma(R) = 0$. Here obviously σ is not of locally finite order.

By the following proposition, we can obtain $W(R[[x]]) = W(R)[[x]]$ for a ring R , which is compared with $W(R[x]) = W(R)[x]$ [2, Corollary 4].

Proposition 3.8. $W(R[[x; \sigma]]) = W(R)[[x; \sigma]]$ when σ is of locally finite order.

Proof. The proof for “ \subseteq ” is similar to one of Theorem 3.6. For the converse, let $g(x) = \sum_{i=0}^m b_i x^i \in W_\sigma(R)[[x; \sigma]]$. Let $\sigma^{s_i}(b_i) = b_i$ and $K_i = \sum_{j=0}^{s_i-1} R\sigma^j(b_i)R$ for $i = 0, 1, \dots$. Then each K_i is σ -invariant and so nilpotent, entailing $K_i[[x; \sigma]] \subseteq W(R[[x; \sigma]])$. Thus $g(x) \in \sum_{i=0}^{\infty} K_i[[x; \sigma]] \subseteq W(R[[x; \sigma]])$. \square

We now turn our attention to observe the Levitzki radical of $R[x; \sigma]$. Note that $L(R) = \{a \in R \mid aR \text{ is locally nilpotent}\}$ with the help of [6, Proposition 10.31]. We can also obtain similar results about the σ -Levitzki radical.

Lemma 3.9.

- (1) $L_\sigma(R)$ is a σ -invariant ideal of R .
- (2)

$$\begin{aligned} L_\sigma(R) &= \left\{ a \in R \mid \sum_{i=0}^{\infty} R\sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent} \right\} \\ &= \left\{ a \in R \mid \sum_{i=m}^{\infty} R\sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent for each } m \in \mathbb{Z} \right\} \\ &= \left\{ a \in R \mid \sum_{i=n}^{\infty} R\sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent for some } n \leq 0 \right\} \end{aligned}$$

$$\begin{aligned}
&= \left\{ a \in R \mid \sum_{i=0}^{\infty} \sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent} \right\} \\
&= \left\{ a \in R \mid \sum_{i=-\infty}^{\infty} R\sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent} \right\}.
\end{aligned}$$

Proof. We apply the proof of Lemma 3.1. It is obvious that S is σ -nilpotent if and only if so is $\sigma^{-1}(S)$ if and only if so is $\sigma(S)$ for a subset S of R . We will use this freely.

(1) Since $\sigma(L_{\sigma}(R)) \subseteq L_{\sigma}(R)$ by the definition, $L_{\sigma}(R) \subseteq \sigma^{-1}(L_{\sigma}(R))$. For the reverse inclusion, let $\sigma^{-1}(a) \in \sigma^{-1}(L_{\sigma}(R))$. Then $a \in \sum_{\text{finite}} I_j$, where each I_j is a locally σ -nilpotent σ -ideal of R . Since I_j is a σ -ideal, $\sigma^{-1}(I_j)$ is a σ -ideal of R by the proof of Lemma 3.1(1). Moreover $\sigma^{-1}(I_j)$ is also locally σ -nilpotent since I_j is locally σ -nilpotent. Thus $\sum_{\text{finite}} \sigma^{-1}(I_j) \subseteq L_{\sigma}(R)$ by Lemma 1.7(1), and this yields $\sigma^{-1}(a) \in L_{\sigma}(R)$. This yields $\sigma^{-1}(L_{\sigma}(R)) \subseteq L_{\sigma}(R)$, entailing $L_{\sigma}(R) \subseteq \sigma(L_{\sigma}(R))$.

(2) Let $A = \{a \in R \mid \sum_{i=0}^{\infty} R\sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent}\}$. Then $A \subseteq L_{\sigma}(R)$ clearly. For the reverse inclusion, let $w \in L_{\sigma}(R)$. Then there exist locally σ -nilpotent σ -ideals I_j in R such that $w \in \sum_{\text{finite}} I_j$. By Lemma 1.7(1), $\sum_{\text{finite}} I_j$ is also a locally σ -nilpotent σ -ideal of R . Thus $\sum_{i=0}^{\infty} R\sigma^i(w)R$ is also locally σ -nilpotent since $\sum_{i=0}^{\infty} R\sigma^i(w)R \subseteq \sum_{\text{finite}} I_j$. Therefore $L_{\sigma}(R) \subseteq A$, entailing $L_{\sigma}(R) = A$.

To obtain the second expression of $L_{\sigma}(R)$, let $a \in L_{\sigma}(R)$. Then, by (1), $\sigma^{-j}(a) \in L_{\sigma}(R)$ for every positive integer j . So the σ -ideal $\sum_{i=0}^{\infty} R\sigma^i(\sigma^{-j}(a))R$ is locally σ -nilpotent by the first expression of $L_{\sigma}(R)$. This yields

$$L_{\sigma}(R) = \left\{ a \in R \mid \sum_{i=m}^{\infty} R\sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent for each } m \in \mathbb{Z} \right\}.$$

Considering the including relation between the first and second expressions, we also obtain

$$L_{\sigma}(R) = \left\{ a \in R \mid \sum_{i=n}^{\infty} R\sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent for some } n \leq 0 \right\}.$$

Next, let $B = \{a \in R \mid \sum_{i=0}^{\infty} \sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent}\}$. If $\sum_{i=0}^{\infty} \sigma^i(a)R$ is locally σ -nilpotent then so is $R(\sum_{i=0}^{\infty} \sigma^i(a)R) = \sum_{i=0}^{\infty} R\sigma^i(a)R$, entailing $a \in L_{\sigma}(R)$ and $B \subseteq L_{\sigma}(R)$. Conversely let $a \in L_{\sigma}(R)$. Then $\sum_{i=0}^{\infty} R\sigma^i(a)R$ is locally σ -nilpotent by the first expression of $W_{\sigma}(R)$, and so $\sum_{i=0}^{\infty} \sigma^i(a)R$ is also locally σ -nilpotent, obtaining $B \supseteq L_{\sigma}(R)$.

Finally, we will show $L_{\sigma}(R) = \{a \in R \mid \sum_{i=-\infty}^{\infty} R\sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent}\}$. To see this, it suffices to show $\sum_{i=-\infty}^{\infty} R\sigma^i(a)R$ is locally σ -nilpotent for any $a \in L_{\sigma}(R)$. Let S be a finite subset of $\sum_{i=-\infty}^{\infty} R\sigma^i(a)R$. Then there exists $n \in \mathbb{Z}$ such that $S \subseteq \sum_{i=n}^{\infty} R\sigma^i(a)R$. By the argument above, $\sum_{i=n}^{\infty} R\sigma^i(a)R$ is locally σ -nilpotent, and so S is σ -nilpotent. \square

The following is similar to Lemma 3.2.

Note. Let I be a right ideal of R with $\sigma(I) \subseteq I$, and $n \geq 1$. Then I is locally σ -nilpotent if and only if I is locally σ^n -nilpotent.

Proof. We apply the proof of Lemma 3.2. Let $l \geq 1$ and S be a finite subset of I . Suppose that I is locally σ -nilpotent. Since I is locally σ -nilpotent, there exists $N = N(nl) \geq 1$ for any $n \geq 1$ such that

$$0 = S\sigma^{nl}(S)\sigma^{2nl}(S)\cdots\sigma^{Nnl}(S) = S\sigma^{nl}(S)\sigma^{n2l}(S)\cdots\sigma^{nNl}(S),$$

entailing that I is locally σ^n -nilpotent.

Conversely suppose that I is locally σ^n -nilpotent right ideal. Then $S\sigma^l(S)\sigma^{2l}(S)\cdots\sigma^{(n-1)l}(S) \subseteq I$. Moreover $S\sigma^l(S)\sigma^{2l}(S)\cdots\sigma^{(n-1)l}(S)$ is also finite. Put $S' = S\sigma^l(S)\sigma^{2l}(S)\cdots\sigma^{(n-1)l}(S)$. Since I is locally σ^n -nilpotent and $S' \subseteq I$ is finite, there exists a positive integer t such that

$$\begin{aligned} 0 &= S'\sigma^{nl}(S')\sigma^{2nl}(S')\cdots\sigma^{tnl}(S') \\ &= (S\sigma^l(S)\sigma^{2l}(S)\cdots\sigma^{(n-1)l}(S))\sigma^{nl}(S\sigma^l(S)\sigma^{2l}(S)\cdots\sigma^{(n-1)l}(S))\cdots \\ &\quad \sigma^{tnl}(S\sigma^l(S)\sigma^{2l}(S)\cdots\sigma^{(n-1)l}(S)) \\ &= S\sigma^l(S)\sigma^{2l}(S)\cdots\sigma^{(n-1)l}(S)\sigma^{nl}(S)\sigma^{(n+1)l}(S)\cdots\sigma^{(tn+n-1)l}(S). \quad \square \end{aligned}$$

The following result can be proved by the same method as in the proof of Lemma 2.1(1).

Lemma 3.10. Let $\bar{R} = R/I$, where I is a locally σ -nilpotent σ -invariant ideal of R . If $\bar{a} \in L_\sigma(\bar{R})$, then $a \in L_\sigma(R)$.

Recall that $x^{-n}a = \sigma^{-n}(a)x^{-n}$ (equivalently, $ax^n = x^n\sigma^{-n}(a)$) for any $n \geq 1$ and $a \in R$.

Theorem 3.11. $L(R[x, x^{-1}; \sigma]) \subseteq (L(R) \cap L_\sigma(R))[x, x^{-1}; \sigma]$.

Proof. Let $S = R[x, x^{-1}; \sigma]$ and $f(x) = \sum_{i=m_1}^{m_2} a_i x^i \in L(S)$ with $m_1 \leq m_2 \in \mathbb{Z}$. Then $Sf(x)S$ is locally nilpotent. Without loss of generality, we can put $f(x) = \sum_{i=0}^m a_i x^i$ with $m \geq 0$. Let $B = \{a_0 r_1, \dots, a_0 r_t\}$ be a finite subset of $a_0 R$ for $r_i \in R$. Then $C = \{f(x)r_1, \dots, f(x)r_t\}$ is a finite subset in $Sf(x)S$. But $Sf(x)S$ is locally nilpotent, and so $C^k = 0$ for some integer $k \geq 2$. Then we have

$$f(x)r_{i_1}f(x)r_{i_2}\cdots f(x)r_{i_k} = 0 \quad (1)$$

where $r_{i_j} \in \{r_1, \dots, r_t\}$. The term of the lowest degree in the equality (1) is

$$a_0 r_{i_1} a_0 r_{i_2} \cdots a_0 r_{i_k}, \quad (2)$$

and so this product (2) is also zero. This yields $B^k = 0$ and so $a_0 \in L(R)$.

Here let $I = \sum_{i=0}^{\infty} \sigma^i(a_0)R$. Let $B_1 = \{b_1, \dots, b_t\}$ be a finite subset of I such that $b_1 = \sum_{i=0}^{n_1} \sigma^i(a_0)r_{1i}, \dots, b_t = \sum_{i=0}^{n_t} \sigma^i(a_0)r_{ti}$ for $r_{ij} \in R$, and $0 \leq n_1 \leq \dots \leq n_t$. Let

$$\begin{aligned} C_1 &= \{x^j f(x)x^{-j}r_j \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\} \\ &= \{\sigma^j(f(x))r_j \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\} \end{aligned}$$

where $\sigma(\sum b_i x^i) = \sum \sigma(b_i)x^i$ (for $\sum b_i x^i \in S$) is the automorphism of S induced by the automorphism σ of R . Let $l \geq 1$ and consider another set

$$\begin{aligned} C_2 &= x^l C_1 = \sigma^l(C_1)x^l = \{x^l \sigma^j(f(x))r_j \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\} \\ &= \{\sigma^l(\sigma^j(f(x))r_j)x^l \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\}. \end{aligned}$$

Then C_2 is also a finite subset of $Sf(x)S$, and so $C_2^h = 0$ for some integer $h \geq 2$ since $Sf(x)S$ is locally nilpotent. Every sum-factor in C_2^h is of the form

$$\begin{aligned}
0 &= (\sigma^l(\sigma^{j_1}(f(x))r_{j_1})x^l)(\sigma^l(\sigma^{j_2}(f(x))r_{j_2})x^l) \cdots (\sigma^l(\sigma^{j_h}(f(x))r_{j_h})x^l) \\
&= \sigma^l(\sigma^{j_1}(f(x))r_{j_1})\sigma^{2l}(\sigma^{j_2}(f(x))r_{j_2})x^{2l} \cdots (\sigma^l(\sigma^{j_h}(f(x))r_{j_h})x^l) \\
&\vdots \\
&= \sigma^l(\sigma^{j_1}(f(x))r_{j_1})\sigma^{2l}(\sigma^{j_2}(f(x))r_{j_2}) \cdots \sigma^{hl}(\sigma^{j_h}(f(x))r_{j_h})x^{hl}
\end{aligned}$$

where $r_{j_q} \in \{r_j \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\}$. Whence

$$\sigma^l(\sigma^{j_1}(f(x))r_{j_1})\sigma^{2l}(\sigma^{j_2}(f(x))r_{j_2}) \cdots \sigma^{hl}(\sigma^{j_h}(f(x))r_{j_h}) = 0. \quad (3)$$

The term of lowest degree in the equality (3) is

$$\sigma^l(\sigma^{j_1}(a_0)r_{j_1})\sigma^{2l}(\sigma^{j_2}(a_0)r_{j_2}) \cdots \sigma^{hl}(\sigma^{j_h}(a_0)r_{j_h}) \quad (4)$$

and so this product (4) is also zero. This yields $B_1\sigma^l(B_1) \cdots \sigma^{hl}(B_1) = 0$ and so $a_0 \in L_\sigma(R)$ by Lemma 3.9(2), entailing $a_0 \in L(R) \cap L_\sigma(R)$.

Note that $R\sigma^i(a_0)R$ is locally nilpotent for any $i \in \mathbb{Z}$ since Ra_0R is locally nilpotent. Moreover, $\sigma^i(a_0) \in L_\sigma(R)$ since $L_\sigma(R)$ is σ -invariant by Lemma 3.9(1). Thus $\sigma^i(a_0) \in L(R) \cap L_\sigma(R)$ and so $\sum_{i=-\infty}^{\infty} R\sigma^i(a_0)R \subseteq L(R) \cap L_\sigma(R)$.

Next let $J = \sum_{i=-\infty}^{\infty} R\sigma^i(a_0)R \subseteq L(R) \cap L_\sigma(R)$ and $S_1 = S/SJS$. Put $f_1(x) = a_1x + \cdots + a_mx^m$. Let $B_2 = \{a_1r_1, \dots, a_1r_t\}$ be a finite subset of a_1R for $r_i \in R$, and $\bar{B}_2 = \{\bar{a}_1\bar{r}_1, \dots, \bar{a}_1\bar{r}_t\}$. Then $C_3 = \{f_1(x)\bar{r}_1, \dots, f_1(x)\bar{r}_t\}$ is a finite subset of $S_1\bar{f}_1(x)S_1$. Since $S_1\bar{f}_1(x)S_1 = S_1\bar{f}_1(x)S_1$ is locally nilpotent in S_1 , $C_3^k = 0$ in S_1 for some $k \geq 2$. This implies $\bar{B}_2^k = 0$ in S_1 , and so $B_2^k \subseteq SJS$. In fact $B_2^k \subseteq RJR = J$. Since J is locally nilpotent by the preceding argument, $(B_2^k)^s = 0$ for some $s \geq 2$, entailing $a_1 \in L(R)$.

We will prove $a_1 \in L_\sigma(R)$. It suffices to show $\bar{a}_1 \in L_\sigma(R/J)$ by Lemma 3.10. Let $\bar{R} = R/J$ and $\bar{J}_1 = \sum_{i=0}^{\infty} \sigma^i(\bar{a}_1)\bar{R}$. Let $B_3 = \{\bar{b}_1, \dots, \bar{b}_t\}$ be a finite subset of \bar{J}_1 such that $\bar{b}_1 = \sum_{i=0}^{n_1} \sigma^i(\bar{a}_1)\bar{r}_{1i}, \dots, \bar{b}_t = \sum_{i=0}^{n_t} \sigma^i(\bar{a}_1)\bar{r}_{ti}$ for $\bar{r}_{ij} \in \bar{R}$, and $0 \leq n_1 \leq \dots \leq n_t$. Let

$$C_4 = \{\sigma^j(\bar{f}_1(x))\sigma^{-1}(\bar{r}_j) \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\}.$$

Let $l \geq 1$ and consider another set

$$\begin{aligned}
C_5 &= x^l C_4 = \sigma^l(C_4)x^l = \{x^l \sigma^j(\bar{f}_1(x))\sigma^{-1}(\bar{r}_j) \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\} \\
&= \{\sigma^l(\sigma^j(\bar{f}_1(x))\sigma^{-1}(\bar{r}_j))x^l \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\}.
\end{aligned}$$

Then C_5 is also a finite subset of $S_1\bar{f}_1(x)S_1$, and so $C_5^h = 0$ in S_1 for some integer $h \geq 2$.

By the preceding argument,

$$\sigma^l(\sigma^{j_1}(\bar{f}_1(x))\sigma^{-1}(\bar{r}_{j_1}))\sigma^{2l}(\sigma^{j_2}(\bar{f}_1(x))\sigma^{-1}(\bar{r}_{j_2})) \cdots \sigma^{hl}(\sigma^{j_h}(\bar{f}_1(x))\sigma^{-1}(\bar{r}_{j_h})) = 0$$

in S_1 . Thus

$$\sigma^l(\sigma^{j_1}(\bar{a}_1)\bar{r}_{j_1})\sigma^{2l}(\sigma^{j_2}(\bar{a}_1)\bar{r}_{j_2}) \cdots \sigma^{hl}(\sigma^{j_h}(\bar{a}_1)\bar{r}_{j_h}) = 0$$

in \bar{R} . This yields $B_3\sigma^l(B_3) \cdots \sigma^{hl}(B_3) = 0$ in \bar{R} and hence $\bar{a}_1 \in L_\sigma(\bar{R})$ by Lemma 3.9(2). Now $a_1 \in L_\sigma(R)$ by Lemma 3.10.

By the same argument as above, $\sum_{i=-\infty}^{\infty} R\sigma^i(a_1)R \subseteq L(R) \cap L_{\sigma}(R)$. Moreover $\sum_{i=-\infty}^{\infty} R\sigma^i(a_0)R + \sum_{i=-\infty}^{\infty} R\sigma^i(a_1)R \subseteq L(R) \cap L_{\sigma}(R)$ by Lemma 1.7(1).

Suppose inductively that $a_i \in L(R) \cap L_{\sigma}(R)$ for $i = 0, 1, \dots, w-1$. Then $\sum_{i=-\infty}^{\infty} R\sigma^i(a_0)R + \dots + \sum_{i=-\infty}^{\infty} R\sigma^i(a_{w-1})R \subseteq L(R) \cap L_{\sigma}(R)$ by the same argument as above. Say $V = \sum_{i=-\infty}^{\infty} R\sigma^i(a_0)R + \dots + \sum_{i=-\infty}^{\infty} R\sigma^i(a_{w-1})R$ and $T = SVS$. Let $S_w = S/T$ and $f_w(x) = \sum_{i=0}^m a_i x^i$. Then $S_w \bar{f}(x) S_w = S_w \bar{f}_w(x) S_w$. Let $B_3 = \{a_w v_1, \dots, a_w v_t\}$ be a finite subset of $a_w S$ for $v_i \in S$ ($i = 1, \dots, t$). Then $C_6 = \{f(x)v_1, \dots, f(x)v_t\}$ is a finite subset in $Sf(x)S$, and so $C_6^k = 0$ for some integer $k \geq 2$. Then $C_6^k = 0$ implies $C_7^k \subseteq T$, where $C_7 = \{f_w(x)v_1, \dots, f_w(x)v_t\}$ is a finite subset in $Sf_w(x)S$. This yields $B_3^k \subseteq V$ by the same argument as above. But B_3^k is also finite and V is locally nilpotent. So B_3^k is also nilpotent, and so $a_w \in L(R)$.

To obtain $a_w \in L_{\sigma}(R)$, it suffices to show $\bar{a}_w \in L_{\sigma}(R/V)$ by Lemma 3.10. Here let $\bar{R} = R/V$ and $\bar{K} = \sum_{i=0}^{\infty} \sigma^i(\bar{a}_w)\bar{R}$. Let $B_4 = \{\bar{b}_1, \dots, \bar{b}_t\}$ be a finite subset of \bar{K} such that $\bar{b}_1 = \sum_{i=0}^{n_1} \sigma^i(\bar{a}_w)\bar{r}_{1i}, \dots, \bar{b}_t = \sum_{i=0}^{n_t} \sigma^i(\bar{a}_w)\bar{r}_{ti}$ for $\bar{r}_{ij} \in \bar{R}$, and $0 \leq n_1 \leq \dots \leq n_t$. Let

$$C_8 = \{\sigma^j(\bar{f}_w(x))\sigma^{-w}(\bar{r}_j) \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\}.$$

Let $l \geq 1$ and consider another set

$$\begin{aligned} C_9 &= x^l C_8 = \sigma^l(C_8)x^l = \{x^l \sigma^j(\bar{f}_w(x))\sigma^{-w}(\bar{r}_j) \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\} \\ &= \{\sigma^l(\sigma^j(\bar{f}_w(x))\sigma^{-w}(\bar{r}_j))x^l \mid j = 1_0, \dots, 1_{n_1}, \dots, t_0, \dots, t_{n_t}\}. \end{aligned}$$

Then C_9 is also a finite subset of $S_w \bar{f}_w(x) S_w$, and so $C_9^h = 0$ in S_w for some integer $h \geq 2$.

Thus, through the same argument as above, we get

$$\sigma^l(\sigma^{j_1}(\bar{f}_w(x))\sigma^{-w}(\bar{r}_{j_1}))\sigma^{2l}(\sigma^{j_2}(\bar{f}_w(x))\sigma^{-w}(\bar{r}_{j_2})) \dots \sigma^{hl}(\sigma^{j_h}(\bar{f}_w(x))\sigma^{-w}(\bar{r}_{j_h})) = 0$$

in S_w . Thus

$$\sigma^l(\sigma^{j_1}(\bar{a}_w)\bar{r}_{j_1})\sigma^{2l}(\sigma^{j_2}(\bar{a}_w)\bar{r}_{j_2}) \dots \sigma^{hl}(\sigma^{j_h}(\bar{a}_w)\bar{r}_{j_h}) = 0$$

in \bar{R} . This yields $B_4 \sigma^l(B_4) \dots \sigma^{hl}(B_4) = 0$ in \bar{R} and hence Lemma 3.9(2) implies $\bar{a}_w \in L_{\sigma}(\bar{R})$. Now $a_w \in L_{\sigma}(R)$ by Lemma 3.10, entailing $a_w \in L(R) \cap L_{\sigma}(R)$. Therefore we have $L(R[x, x^{-1}; \sigma]) \subseteq (L(R) \cap L_{\sigma}(R))[x, x^{-1}; \sigma]$. \square

By Note after Proposition 3.17 below, $L(R[x, x^{-1}; \sigma]) = (L(R) \cap L_{\sigma}(R))[x, x^{-1}; \sigma] = L(R)[x, x^{-1}; \sigma] = L_{\sigma}(R)[x, x^{-1}; \sigma]$ when R is left or right Noetherian. In general, $L(R) \neq L_{\sigma}(R)$ by Example 3.14 below.

Lemma 3.12. $L(R) = L_{\sigma}(R)$ when σ is of locally finite order.

Proof. Letting $a \in L_{\sigma}(R)$ and $\sigma^k(a) = a$, then RaR is locally σ -nilpotent. Note that every σ -nilpotent finite subset of R is nilpotent and so $L_{\sigma}(R) \subseteq L(R)$. Conversely, let $a \in L(R)$. Then aR is locally nilpotent. Note that $\sigma^i(aR) = \sigma^i(a)R$ is locally nilpotent for each integer $i \geq 0$. Since $\sigma^k(a) = a$, $\sum_{i=0}^{k-1} R\sigma^i(a)R = \sum_{i=0}^{k-1} R\sigma^i(a)R$ is a locally nilpotent σ -ideal. Let $S = \{\sum_{\text{finite}} \sigma^j(a)r \mid j = 0, 1, \dots, k-1\}$ be a finite subset of $\sum_{i=0}^{k-1} \sigma^i(a)R$. For each r , we let $\sigma^{k_r}(r) = r$. Consider $S' = \{\sum_{\text{finite}} \sigma^i(a)\sigma^t(r) \mid \sum_{\text{finite}} \sigma^j(a)r \in S\}$, where $i \in \{0, 1, \dots, k-1\}$ and $t \in \{0, 1, \dots, k_r-1\}$. Notice that $\sigma^u(S') \subseteq S'$ for all $u \geq 1$ and $|S'| < \infty$. Then $S'^n = 0$ for some $n \geq 2$. Therefore for any integer $l \geq 1$, $S\sigma^l(S) \dots \sigma^{(n-1)l}(S) \subseteq S'\sigma^l(S') \dots \sigma^{(n-1)l}(S') \subseteq S'^n = 0$. \square

Like the Wedderburn radical of $R[x; \sigma]$, $L(R[x; \sigma]) \neq L_{\sigma}(R)[x; \sigma]$ by Example 3.14 below. However we have the following.

Theorem 3.13. $L(R[x; \sigma]) = L(R)[x; \sigma]$ when σ is of locally finite order.

Proof. Let $T = R[x; \sigma]$ and $f(x) = \sum_{i=0}^m a_i x^i \in L(T)$. Then $f(x)T$ is locally nilpotent in T . Note that σ induces an automorphism of $R[x; \sigma]$ via $\sigma(f(x)) \mapsto \sum_{i=0}^m \sigma(a_i) x^i$, and we again denote this automorphism by σ . By Lemma 3.9(2), we first claim that $\sum_{i=0}^\infty \sigma^i(a_0)R = \sum_{i=0}^{k_0-1} \sigma^i(a_0)R$ is locally σ -nilpotent in R , where $\sigma^{k_0}(a_0) = a_0$. Let $S = \{\sum_{i=0}^{k_0-1} \sigma^i(a_0)r_{i_h} \mid i_h = 0, 1, 2, \dots, k_0 - 1\}$ be a finite subset of $\sum_{i=0}^{k_0-1} \sigma^i(a_0)R$. Consider $S' = \{\sum_{i=0}^{k_0-1} \sigma^i(a_0)\sigma^{j_0}(r_{i_h}) \mid \sum_{i=0}^{k_0-1} \sigma^i(a_0)r_{i_h} \in S\}$, where $0 \leq i_h \leq k_0 - 1$ and j_0 runs in the set $A = \bigcup_{i_h} A_{i_h}$ with $A_{i_h} = \{0, 1, \dots, k_{i_h} - 1 \mid \sigma^{k_{i_h}}(r_{i_h}) = r_{i_h}\}$. Note here that A is finite since $\{r_{i_h}\}$ is finite. Then $\sigma^u(S') \subseteq S'$ and $\sigma^u(S) \subseteq S'$ for all $u \geq 0$, moreover $|S'| < \infty$. Fix an integer $l \geq 1$. We will show that $S'\sigma^l(S') \cdots \sigma^{sl}(S') = 0$ for some positive integer s . Then clearly $S\sigma^l(S) \cdots \sigma^{sl}(S) = 0$. Now consider the subset

$$V = \left\{ \sum_{\text{finite}} \sigma^{i_h}(f(x))\sigma^{j_0}(r_{i_h}) \mid \sum_{\text{finite}} \sigma^{i_h}(a_0)r_{i_h} \in S \right\}$$

of $Tf(x)T$. Since $Tf(x)T$ is also locally nilpotent and $|V| < \infty$, we get $V^{t_0} = 0$ for some $t_0 \geq 2$, entailing $(S')^{t_0} = 0$. Since $\sigma^u(S') \subseteq S'$ for all $u \geq 1$, we obtain

$$S'\sigma^l(S') \cdots \sigma^{(t_0-1)l}(S') = 0.$$

This yields $a_0 \in L_\sigma(R) = L(R)$.

Next note that $\sigma^i(a_0) \in L_\sigma(R)$ for any $i \in \mathbb{Z}$ since $L_\sigma(R)$ is σ -invariant by Lemma 3.9(1). Thus $\sum_{i=-\infty}^\infty R\sigma^i(a_0)R \subseteq L_\sigma(R) = L(R)$. Let $I = \sum_{i=-\infty}^\infty R\sigma^i(a_0)R$ and consider $\bar{T} = T/TIT$. Then since $f(x)T$ is locally nilpotent in T , $\bar{f}(x)\bar{T}$ is also locally nilpotent in \bar{T} . We identify polynomials of T with their images in \bar{T} , and so we may let $\bar{f}_1(x) = a_1x + \cdots + a^m x^m$. Put $S_1 = \{\sum_{i=0}^{k_1-1} \sigma^i(a_1)s_{i_h} \mid i_h = 0, 1, \dots, k_1 - 1\}$ be a finite subset of $\sum_{i=0}^{k_1-1} \sigma^i(a_1)R$, where $\sigma^{k_1}(a_1) = a_1$. Let $S'_1 = \{\sum_{i=0}^{k_1-1} \sigma^i(a_1)\sigma^{j_1}(s_{i_h}) \mid \sum_{i=0}^{k_1-1} \sigma^i(a_1)s_{i_h} \in S_1\}$, where $0 \leq i \leq k_1 - 1$ and j_1 runs in the set $B = \bigcup_{i_h} B_{i_h}$ with $B_{i_h} = \{0, 1, \dots, v_{i_h} - 1 \mid \sigma^{v_{i_h}}(s_{i_h}) = s_{i_h}\}$. Note here that B is finite since $\{s_{i_h}\}$ is finite. Then $\sigma^u(S'_1) \subseteq S'_1$ and $\sigma^u(S_1) \subseteq S'_1$ for all $u \geq 0$, moreover $|S'_1| < \infty$. By the preceding argument, $(S'_1)^{t_1} \subseteq TIT$ for some $t_1 \geq 2$. Note that $(S'_1)^{t_1} \subseteq RIR = I$. Since I is locally nilpotent, $(S'_1)^{t_2} = 0$ for some $t_2 \geq 2$. Hence we have

$$S'_1\sigma^l(S'_1) \cdots \sigma^{(t_2-1)l}(S'_1) = 0,$$

entailing $a_1 \in L_\sigma(R)$.

By the same argument as above, $J = \sum_{i=-\infty}^\infty R\sigma^i(a_0)R + \sum_{j=-\infty}^\infty R\sigma^j(a_1)R$ is a locally σ -nilpotent ideal of R . Consider $T/(TIT + TJT)$. Proceeding in this manner, we finally obtain $a_0, \dots, a_m \in L_\sigma(R) = L(R)$ inductively.

For the converse, it suffices to show that $f(x)T$ is locally nilpotent for any $f(x) = \sum_{i=0}^m a_i x^i \in L(R)[x; \sigma] = L_\sigma(R)[x; \sigma]$. Let $S = \{f(x)g_i(x), \dots, f(x)g_t(x) \mid g_i(x) \in T\}$ be a finite subset of $f(x)T$, and let $g_i(x) = b_{i_0} + b_{i_1}x + \cdots + b_{i_{n_i}}x^{n_i}$. Consider the set

$$S' = \{\sigma^l(a_w \sigma^w(b_{i_j})) \mid w = 0, 1, \dots, m \text{ and } l = 0, 1, \dots, t_{w,i_j}\},$$

where t_{w,i_j} is the order of $a_w \sigma^w(b_{i_j})$. Since $\sigma^l(a_w) \in L_\sigma(R) = L(R)$ for any l and w , we get S' is a finite subset of $L(R)$, and so S' is nilpotent since $L(R)$ is the largest locally nilpotent ideal of R . Therefore S is a nilpotent subset of T since every sum-factor of every coefficient of a polynomial in all powers of S is contained in the power of S' , completing the proof. \square

The following example shows that the condition “ σ is of locally finite order” is not superfluous in Lemma 3.12 and Theorem 3.13. Moreover, it shows that $L(R[x; \sigma]) \neq L_\sigma(R)[x; \sigma]$.

Example 3.14. Let K be an infinite field and $K[\{t_i\}_{i \in \mathbb{Z}}]$ the polynomial ring over K , and $J = \langle \{t_{n_1} t_{n_2} t_{n_3} \mid n_3 - n_2 = n_2 - n_1 > 0\} \rangle$ be the ideal of $K[\{t_i\}_{i \in \mathbb{Z}}]$. Define $R = K[\{t_i\}_{i \in \mathbb{Z}}]/J$. The K -automorphism σ of $K[\{t_i\}_{i \in \mathbb{Z}}]$ defined by sending each t_i to t_{i+1} induces an automorphism σ on R . Note that σ is not of locally finite order.

Since t_1 is not nilpotent, $t_1 R$ is not locally nilpotent. Thus $t_1 \notin L(R)$, but $t_1 \in L_\sigma(R)$ referring Example 1.9. Hence $L(R) \neq L_\sigma(R)$. This implies that the condition “ σ is of locally finite order” in Lemma 3.12 is essential. Moreover, since $t_1 x R[x; \sigma]$ is a locally nilpotent ideal, $L(R[x; \sigma]) \neq L(R)[x; \sigma]$.

Note that $N(R[x; \sigma]) \neq N_\sigma(R)[x; \sigma]$ in general, where $N(R[x; \sigma])$ denotes the upper nilradical of $R[x; \sigma]$. By the same method as in the proof of Lemmas 3.1(2) and 3.9(2), we have $N_\sigma(R) = \{a \in R \mid \sum_{i=0}^{\infty} R \sigma^i(a) R \text{ is } \sigma\text{-nil}\}$.

Theorem 3.15. $N(R[x, x^{-1}; \sigma]) \subseteq (N(R) \cap N_\sigma(R))[x, x^{-1}; \sigma]$.

Proof. Let $T = R[x, x^{-1}; \sigma]$ and $f(x) = \sum_{i=m_1}^{m_2} a_i x^i \in N(T)$ with $m_1 \leq m_2 \in \mathbb{Z}$. Then $Tf(x)T$ is a nil ideal of T . Without loss of generality, we can put $f(x) = \sum_{i=0}^m a_i x^i$ with $m \geq 0$. Note that $T \sigma^j(f(x))T$ is also nil for any $j \geq 0$. We first claim $a_0 \in N(R)$. Let $\sum_{\text{finite}} r a_0 s \in R a_0 R$. Note that $\sum_{\text{finite}} r a_0 s + \sum_{\text{finite}} r a_1 \sigma(s) x + \cdots + \sum_{\text{finite}} r a_m \sigma^m(s) x^m \in Tf(x)T$. Thus $(\sum_{\text{finite}} r a_0 s)^k = 0$ for some integer $k \geq 2$, entailing $R a_0 R$ is nil. We next claim $a_0 \in N_\sigma(R)$. Let $\sum_{\text{finite}} r \sigma^j(a_0) s \in \sum_{i=0}^{\infty} R \sigma^i(a_0) R$. Fix an integer $l \geq 1$. Then

$$\sum_{\text{finite}} r \sigma^j(a_0) \sigma^l(\sigma^{-l}(s)) x^l + \cdots + \sum_{\text{finite}} r \sigma^j(a_m) \sigma^{m+l}(\sigma^{-l}(s)) x^{m+l} \in \sum_{\text{finite}} T \sigma^j(f(x)) x^l T.$$

Put $\alpha = \sum_{\text{finite}} r \sigma^j(a_0) s$. Since $\sum_{\text{finite}} T \sigma^i(f(x)) x^l T$ is nil, $\alpha \sigma^l(\alpha) \cdots \sigma^{nl}(\alpha) = 0$. This implies that $\sum_{i=0}^{\infty} R \sigma^i(a_0) R$ is σ -nil. Therefore $a_0 \in N(R) \cap N_\sigma(R)$.

To show $a_1 \in N(R) \cap N_\sigma(R)$, we consider $\tilde{T} = T/TIT$, where $I = \sum_{i=0}^{\infty} R \sigma^i(a_0) R$. Since $Tf(x)T$ is nil in T , $\tilde{T} \tilde{f}(x) \tilde{T}$ is nil in \tilde{T} . We identify polynomials of T with their images in \tilde{T} , and so we may let $\tilde{f}(x) = a_1 x + \cdots + a_m x^m$. Then $\tilde{T} \tilde{f}(x) \tilde{T} = \tilde{T}(a_1 x + \cdots + a_m x^m) x^{-1} \tilde{T} = \tilde{T}(a_1 + \cdots + a_{m-1} x^{m-1}) \tilde{T}$. By the same method as above, $R a_1 R$ is nil. Let $\beta = \sum_{\text{finite}} r \sigma^j(a_1) s \in \sum_{i=0}^{\infty} R \sigma^i(a_1) R$. Since $\tilde{T}(a_1 + \cdots + a_{m-1} x^{m-1}) x^l \tilde{T}$ is nil for an integer $l \geq 1$, $\beta \sigma^l(\beta) \cdots \sigma^{nl}(\beta) \in I$ for some $n \geq 1$. Put $\gamma = \beta \sigma^l(\beta) \cdots \sigma^{nl}(\beta)$. Since I is σ -nil, $\gamma \sigma^{(n+1)l}(\gamma) \cdots \sigma^{pl}(\gamma) = 0$ for some $p \geq 2$, entailing $a_1 \in N_\sigma(R)$ by Lemma 3.1. Continuing this process, we get $a_0, a_1, \dots, a_m \in N(R) \cap N_\sigma(R)$, completing the proof. \square

Theorem 3.16. $N(R[x; \sigma]) \subseteq N_\sigma(R)[x; \sigma]$.

Proof. Let $T = R[x; \sigma]$ and $f(x) = \sum_{i=0}^m a_i x^i \in N(T)$. Then $Tf(x)T$ is a nil ideal of T and $T \sigma^j(f(x))T$ is also nil for any $j \geq 0$. By the same method as in the proof of Theorem 3.15, we get $a_0 \in N(R) \cap N_\sigma(R)$.

Next we claim $a_1 \in N_\sigma(R)$. Consider $\tilde{T} = T/TIT$, where $I = \sum_{i=0}^{\infty} R \sigma^i(a_0) R$. Since $Tf(x)T$ is nil in T , $\tilde{T} \tilde{f}(x) \tilde{T}$ is nil in \tilde{T} , and so is $\tilde{T} \sigma^j(\tilde{f}(x)) \tilde{T}$ for any $j \geq 0$. Let $\sum_{\text{finite}} r \sigma^i(a_1) s \in \sum_{i=0}^{\infty} R \sigma^i(a_1) R$ and fix an integer $l \geq 1$. Then

$$\sum_{\text{finite}} \tilde{r} \sigma^i(\tilde{a}_1) \sigma^l(\sigma^{-l}(\tilde{s})) x^l + \cdots + \sum_{\text{finite}} \tilde{r} \sigma^i(\tilde{a}_m) \sigma^{m+l-1}(\sigma^{-l}(\tilde{s})) \tilde{s} x^{m+l-1} \in \sum_{\text{finite}} \tilde{T} \sigma^i(\tilde{f}(x)) x^{l-1} \tilde{T}.$$

Put $\alpha = \sum_{\text{finite}} r\sigma^i(a_1)s$. Since $\sum_{\text{finite}} \bar{T}\sigma^i(\bar{f}(x))x^{l-1}\bar{T}$ is nil in \bar{T} , $\alpha\sigma^l(\alpha)\sigma^{2l}(\alpha) \cdots \sigma^{ul}(\alpha) \in I$ for some $u \geq 2$. Letting $\beta = \alpha\sigma^l(\alpha)\sigma^{2l}(\alpha) \cdots \sigma^{ul}(\alpha)$, then $\beta\sigma^{(u+1)l}(\beta) \cdots \sigma^{ql}(\beta) = 0$ for some $q \geq 3$. Hence α is σ -nilpotent and so $\sum_{i=0}^{\infty} R\sigma^i(a_1)R$ is σ -nil, entailing $a_1 \in N_{\sigma}(R)$. By the preceding argument, we can get that $\sum_{i=0}^{\infty} R\sigma^i(a_2)R$ is σ -2-nil and so is σ -nil by [7, Proposition 3.7]. Continuing this process, we have $a_0, a_1, \dots, a_m \in N_{\sigma}(R)$. \square

Proposition 3.17. $N(R) = N_{\sigma}(R)$ when σ is of locally finite order.

Proof. Let $a \in N_{\sigma}(R)$. Then RaR is σ -nil. Let $\alpha \in RaR$ and $\sigma^k(\alpha) = \alpha$. Since RaR is σ -nil, α is nilpotent and so RaR is nil, entailing $a \in N(R)$. Conversely, let $a \in N(R)$. Then RaR is nil, and so $R\sigma^i(a)R$ is nil for any integer $i \geq 1$. This implies that $\sum_{i=0}^{k-1} R\sigma^i(a)R$ is a nil σ -ideal of R , where $\sigma^k(a) = a$. Let $\alpha \in \sum_{i=0}^{k-1} R\sigma^i(a)R$. Then for any integer $l \geq 1$, $\alpha\sigma^l(\alpha)\sigma^{2l}(\alpha) \cdots \sigma^{(k-1)l}(\alpha) \in \sum_{i=0}^{k-1} R\sigma^i(a)R$. Since $\sum_{i=0}^{k-1} R\sigma^i(a)R$ is nil, $(\alpha\sigma^l(\alpha)\sigma^{2l}(\alpha) \cdots \sigma^{(k-1)l}(\alpha))^t = 0$ for some integer $t \geq 2$. Note that $\sigma^{nl}(\alpha) = \sigma^{(pk+n)l}(\alpha)$ for any integer $p \geq 1$, where $0 \leq n \leq k-1$. Thus

$$\alpha\sigma^l(\alpha)\sigma^{2l}(\alpha) \cdots \sigma^{(k-1)l}(\alpha)\sigma^{kl}(\alpha) \cdots \sigma^{((t-1)k+(k-1))l}(\alpha) = 0,$$

and so α is σ -nilpotent. Therefore $\sum_{i=0}^{k-1} R\sigma^i(a)R$ is a σ -nil σ -ideal of R , entailing $a \in N_{\sigma}(R)$. \square

By Theorem 3.16 and Proposition 3.17, we have $N(R[x; \sigma]) \subseteq N(R)[x; \sigma]$ when σ is of locally finite order.

Note. (1) If R satisfies the ascending chain condition on left annihilators, then $P_{\sigma}(R) = L_{\sigma}(R) = N_{\sigma}(R)$.

(2) If R is a left or right Noetherian ring, then

$$W(R) = P(R) = L(R) = N(R) = W_{\sigma}(R) = P_{\sigma}(R) = L_{\sigma}(R) = N_{\sigma}(R)$$

and

$$N(R[x, x^{-1}; \sigma]) = N_{\sigma}(R)[x, x^{-1}; \sigma] = W_{\sigma}(R)[x, x^{-1}; \sigma] = W(R[x, x^{-1}; \sigma]).$$

Proof. (1) One can compare this proof with one of [7, Proposition 3.14]. It suffices to show $N_{\sigma}(R) \subseteq P_{\sigma}(R)$, considering [12, Proposition 3.9]. Suppose that $a \in N_{\sigma}(R)$ with $\ell(a)$ as large as possible, where $\ell(a)$ denotes the left annihilator of a . Assume $a \notin P_{\sigma}(R)$. Then, by Proposition 1.3, $aR\sigma^n(a) \neq 0$ for some n , and so there exists $b \in R$ such that $ab\sigma^n(a) \neq 0$, entailing $\ell(a) = \ell(ab\sigma^n(a))$. Put $b = \sigma^n(b')$ for some $b' \in R$. Then $0 \neq b\sigma^n(a) = \sigma^n(b'a)$ and so $b' \notin \ell(a) = \ell(ab\sigma^n(a))$. This implies $b'ab\sigma^n(a) \neq 0$ and $\sigma^{-n}(b'ab) \notin \ell(a) = \ell(ab\sigma^n(a))$. Thus $0 \neq b'ab\sigma^n(ab\sigma^n(a)) = b'a\sigma^n(b'a)\sigma^{2n}(a)$. Continuing in this fashion, we get that $b'a$ is not σ -nilpotent, contradicting to $b'a \in N_{\sigma}(R)$.

(2) Let R be a left or right Noetherian ring. Then $N(R)$ is nilpotent by [8], entailing $N(R) = W(R)$. Moreover, we have $W(R) = W_{\sigma}(R)$ by Note after Corollary 3.4. Next $P_{\sigma}(R) = P(R)$ by [7, Corollary 3.13], entailing $W(R) = P(R) = L(R) = N(R) = W_{\sigma}(R) = P_{\sigma}(R) = L_{\sigma}(R) = N_{\sigma}(R)$.

Finally, since R is a left or right Noetherian ring, $N_{\sigma}(R)$ is nilpotent, and so $N_{\sigma}(R)[x, x^{-1}; \sigma] \subseteq N(R[x, x^{-1}; \sigma])$. Thus $N(R[x, x^{-1}; \sigma]) = N_{\sigma}(R)[x, x^{-1}; \sigma]$ by Theorem 3.15. \square

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